Rational Billiards

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Abstract

In this article we seek to provide some insight into the study of rational billiards. We start by discussing the idea that the trajectory of a particle that we wish to track inside of a k-gon can be more easily expressed as a line in the flat plane which is tiled by the afforementioned k-gons. We also discuss the idea of invariant measures and ergodicity as it relates to billiards.

1 Introduction and The General Billiards Case

Let's start by defining our billiards table, let's call the boundary of the table a curve γ, a smooth and closed curve. Let's also define a space M of all the unit tangent vectors with their feet on γ and pointing inwards the convex polygon. Thus we now have a set of starting points for out billiards ball, an element of M, a vector (x, v). Suppose now that we let the billiards ball go, and it hits γ again at x_1, with a new velocity vector v_1, we can now define a transformation $T : M \to M$ that takes in (x, v) and outputs its reflection across the boundary of γ, (x_1, v_1).
We are now ready to define our measure, in this case an area measure. We want to parameterize γ by arc length t (perimeter) and the angle α between v and a side of γ. Now, we can notice that (t, α) describe M. We denote the area measure μ as:

$$\mu = \sin(\alpha)\frac{\delta\alpha}{\delta t}$$

which we claim is invariant with respect to the function T.

Proof. Note that $sin\alpha > 0$ on M. To show its invariance, let $d(t, t_1)$ be the distance between the points γ_t and γ_{t_1}. Then we notice that the partial derivative $\frac{\partial f}{\partial t}$ is the projection of the gradient of $\gamma_t\gamma_{t_1}$ onto the curve at point γ_t. This gradient makes angle α with the curve, so $\frac{\partial f}{\partial t} = \cos\alpha$. Similarly, $\frac{\partial f}{\partial t_1} = -\cos\alpha_1$. Therefore,

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial t_1}dt_1 = -\cos\alpha_1 + \cos\alpha$$

Hence,

$$0 = d^2 f = \sin(\alpha)\frac{\delta\alpha}{\delta t} - \sin(\alpha)\frac{\delta\alpha}{\delta t_1}$$

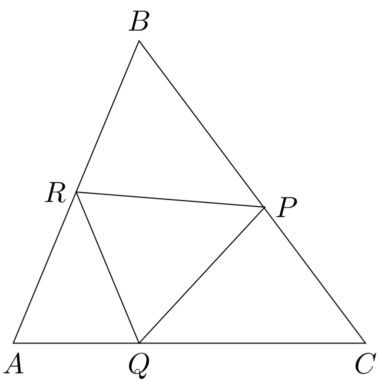
□

2 Trajectories in Polygons

2.1 The Fagano Construction

Let's start with the simple case of an acute triangle. Then there is a periodic trajectory given by the following elementary geometric construction due to Fagano:

Lemma 2.1. *The triangle connecting the base of the altitudes is a 3 periodic billiards trajectory.*



Proof.
The quadrilateral $BPOR$ has two right angles, thus it is inscribed in a circle. Now, since the angles APR and ABQ subtend the same arc, those angles are equal, and similarly, the angles APQ and ACR are equal. Now it just remains to show that ABQ and ACR are equal, and this is shown since they both complement BAC, and so the result follows that they are periodic as all the angles of incidences are equal to the angle of reflection. □

Indeed, the Fagano Construction degenerates for right triangles, in which case there does exist constructions that yield periodic trajectories.

2.2 The Poincare Recurrence Theorem

The Poincare Recurrence Theorem is stated as follows:

Theorem 2.2 (Poincaré Recurrence Theorem)**.** *Let* $(X, , \mu)$ *be a finite measure space, and let* T *be a measure-preserving transformation. Then* T *is recurrent.*

In other words, if T is a measure preserving transformation on a finite measure space. Then, for any set A there exists a point $x \in A$ that returns to A, and the set of points in A that return to A has measure 0.

2.3 Diving into Periodic Trajectories in Rational Polygons

A polygon is called rational if all its angles are rational multiples of π. The first critical observation is that a trajectory can only have finitely many directions, so we must introduce a group $G(P)$ to be able to keep track of them for our polygon P. Now to generate $G(P)$, we draw the parallel line through the center to each of the sides, and then let $G(P)$ be the group of linear isometries generated by the reflections of those lines. Thus, when a ball bounces of a side, it is acted on by a member of $G(P)$. Now, we let the angles of our polygon be $\frac{m_i \pi}{n_i}$ with m_i, n_i coprime integers. Now let N be the least common multiple of all the n_i. Then $G(P)$ is just the group of of symmetries of a regular N-gon,which has $2N$ elements. Thus the maximum number of directions a billiard ball can go is $2N$.

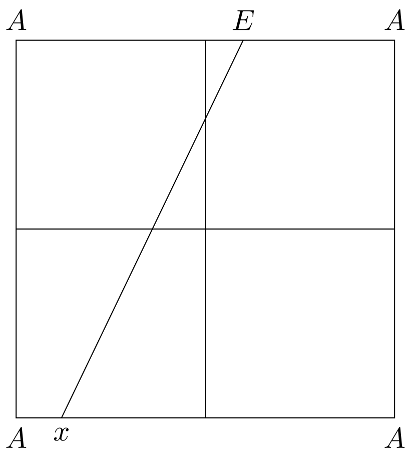
The two dimensional space splits into many invariant one dimensional subspaces which we can relate to the different directions of billiards trajectories. As such, each subspace has an invariant length element, which we can visualize as the width a parallel beam of rays of the trajectory.
In this way, we can construct very specific types of periodic trajectories in rational polygons. The process is as follows: We first choose a side s, and we let U consist of unit vectors with a foot on s and orthagonal to s. Then by Poincare's Recurrence Theorem, there is a phase point in U that at some point, returns to U. The respective starts from side s and returns at a right angle as well, at which point in just repeats its trajectory backwars, and is this periodic.
Rational Polygons are in fact the only group of shapes for which billiards trajectories are well characterized and understood. For example, we can find more periodic trajectories in right triangles:

Theorem 2.3. *When given a right triangle, a.e billiard trajectory that begins at the side of the right angle in a orthogonal direction returns to this side in the same direction.*

Proof. We already know this from above for rational right triangles. For irrational triangles, we must do something different. We tile the plane with rhombi that consist of 4 of these triangles joined at their right angle, and we let α be the acute angle of the rhombus. The trajectories are now just lines in the plane tiled by these rhombi.
We refer to the original rhombus as R_0. Notice that each time we reflect a rhombus in this construction, we rotate it by angle $\pm\alpha$. Then, up to translations, the rhombi can be indexed by R_n with $n \in \mathbb{Z}$.
We now do the following construction. We use 4 of these rhombi in consecutive order and make them a kinds of torus shape by which we loop back from the last to the first, from which we get an infinite surface of these rhombi partially foliated by the billiard beam. A trajectory in R_n may either intersect R_{n-1} in which case we call this a negative intersection, or R_{n+1} which we call a positive intersection.
We want to show that almost all trajectories return to R_0. Since the triangle is irrational, for every $\epsilon > 0$ there exists an $n > 0$ such that the vertical projection to the positive intersecting side is less than ϵ, since irrational rotations are ergodic, so the set of trajectories that reach R_{n+1} has measure less than ϵ. Since ϵ was arbitrarily small, the result is shown. □

2.4 Ergodic Theory

Consider billiards inside a unit square. By tiling the plane with such squares, we reduce the billiard trajectory to a line in that plane. Now, if we consider a 2x2 square that consists of 4 unit squares that share a vertex, then we connect its oppsite sides to make a torus, similarly to the triangle above. The billiard trajectory is just a geodesic line on that torus.



Consider the trajectories at an angle θ from a point x from the lower side of the square. This trajectory intersects the upper side of the square at a point $x + 2\cot\theta \pmod 2$. Then when we rescale into the smaller square, we get the rotational transformation T which takes $x \to x + \cot\theta$ (mod 1). Thus the billiard trajectory in a fixed direction reduces to a circle rotation.
From our ergodic properties of rotations, if the slope of the trajectory is rational, we know that it is periodic, and if it is irrational, then the rotation is ergodic(in fact, uniquely ergodic) and is dense everywhere and uniformly distributed in the square.
It is not known whether every polygon has a periodic billiard trajectory; this is unknown even for obtuse triangles. Substantial progress has recently been made by R. Schwartz, who proved that every obtuse triangle with angles not exceeding 100 degrees has a periodic billiard path. This work significantly relies on a computer program, however.

2.5 Conclusion and Application

Billiards have applications in various different classes of theory ranging from physics to chemistry. The techniques used, especially for rational polygons, are not only efficient but very reproducible, which makes for the rapid development in theory in the field. As of now, there is a ton of new work yet to be done, but the rate at which it is being done is astounding and hard to keep up with.

References

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