A JOURNEY THROUGH TOPOLOGICAL GROUPS

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Abstract. In this paper, we will explore the foundations of topology. We will build knowledge starting with basic point-set topology and ending with the theory of topological groups. Along the way, the reader will no doubt pick up valuable skills and techniques in analysis.

To begin with, we will introduce the foundational definitions in topology. Then, we introduce some particularly important topologies, such as the product topology, and in a later section, the quotient topology.

Next, we come upon deeper topological properties, such as the separation axioms. These allow us to characterize topological spaces, which proves to be useful in the later sections of the paper. We also introduce the notion of continuity from the topological standpoint by looking at maps from one topological space to another.

Finally, we touch upon topological groups, the main topic of the paper. We investigate special properties of topological groups and delve into their separation properties. We briefly talk about quotient spaces and locally compact groups. Then, we part with some concluding remarks.

1. Some Point-Set Topology

We start out with the basics definitions and theorems of topology that we will use later. We will state some of this background material without proof and focus our energy on proving theorems in later sections.

Definition 1.1. A topology on a set X is a collection τ of subsets of X such that
(i) $\emptyset$ and X are in τ,
(ii) The union of any elements in τ is in τ,
(iii) and the intersection of any finite subcollection of τ is in τ.

Definition 1.2. We call a set $U \subseteq X$ open if $U \in \tau$.

Definition 1.3. If τ_1 and τ_2 are two topologies on X, we say that τ_1 is finer than τ_2 if $\tau_2 \subset \tau_1$. In this case, we also say that τ_2 is coarser than τ_1.

Now, we introduce a tool that is particularly useful in characterizing a given topology.

Definition 1.4. For a set X, we define a basis $\mathcal{B}$ of a topology on X to be a collection of subsets of X (called basis elements) such that
(i) For each $x \in X$, there is at least one basis element B containing x, and
(ii) If x belongs to the intersection of two basis elements $B_1 \cap B_2$, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, we define the topology τ generated by $\mathcal{B}$ to be as follows: a subset U of X is said to be open (i.e. an element of τ) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Notice that each basis element is itself a member of the topology.

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Theorem 1.5. *The topology generated by $\mathcal{B}$ is indeed a topology, i.e., it satisfies the properties of a topology listed in Definition 1.1.*

Proof. This easily follows from showing closure under unions and intersections of elements of the topology. ■

Proposition 1.6. *τ is equal to the collection of all unions of elements of the basis $\mathcal{B}$.*

Proof. Any collection of elements of $\mathcal{B}$ is in τ. Since τ is a topology, their union is in τ as well. Conversely, given $U \in \tau$, choose for each $x \in U$ a basis element $B_x \in \mathcal{B}$ for which $x \in B_x \subset U$ (by definition of a basis element). Then

$$U = \bigcup_{x \in U} B_x$$

so U is a union of elements of $\mathcal{B}$. ■

Theorem 1.7. *Let X be a topological space. Let $\mathcal{C}$ be a collection of open sets of X such that for each open set $U \subseteq X$ and each $x \in X$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology on X.*

2. A Glimpse into the Product Topology

We now come to a type of topology that will come in handy later: the product topology.

Definition 2.1. Let X and Y be two topological spaces. The product topology on $X \times Y$ is defined as the topology generated by the basis $\mathcal{B}$, where $\mathcal{B}$ is the collection of sets of the form $U \times V$, for open subsets U and V of X and Y, respectively.

To check that $\mathcal{B}$ is indeed a basis, first note that $X \times Y$ is a basis element. To verify the second condition, observe that the intersection of two basis elements is another basis element; for basis elements $U_1 \times V_1$ and $U_2 \times V_2$,

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \times V_2)$$

and the right hand side shows that the intersection is a basis element since $U_1 \cap U_2$ and $V_1 \cap V_2$ are open sets.

Let us now characterize the basis of the product topology in terms of the bases of X and Y.

Theorem 2.2. *If $\mathcal{C}$ is a basis of the topology on X and $\mathcal{D}$ is a basis of the topology on Y, then*

$$\mathcal{B} = \{C \times D \mid C \in \mathcal{C} \text{ and } D \in \mathcal{D}\}$$

is a basis for the topology on $X \times Y$.

Proof. We will use Theorem 1.7. Given an open set W of $X \times Y$, and a point $x \times y \in W$, by definition of the product topology there is a basis element $U \times V$ such that $x \times y \in U \times V \subset W$. Because $\mathcal{C}$ and $\mathcal{D}$ are bases for X and Y, respectively, we must have basis elements $C \in \mathcal{C}$ and $D \in \mathcal{D}$ such that $x \in C \subset U$ and $y \in D \subset V$. Then, $x \times y \in C \times D \subset W$, as desired. We conclude that $\mathcal{B}$ is indeed a basis for $X \times Y$. ■

3. Closed Sets, Limit Points, and the Hausdorff Axiom

Let us start out simple:

Definition 3.1. Call a subset $A \subseteq X$ closed if the set $X - A$ is open.

The notion of a closed set actually leads to some nice properties.

Proposition 3.2. *For a topological space X, the following properties hold:*
(i) $\emptyset$ and X are closed
(ii) Arbitrary intersections of closed sets are closed
(iii) Finite unions of closed sets are closed.

Proof. (i) This follows trivially from taking the complements of the sets X and $\emptyset$, respectively.
(ii) Given a collection of closed sets $\{A_\alpha\}$ for $\alpha \in J$ (for some arbitrary set J), we can use De Morgan's Law to obtain

$$X - (\bigcap_{\alpha \in J} A_\alpha) = \bigcup_{\alpha \in J} (X - A_\alpha)$$

Since the sets $X - A_\alpha$ are open, so is their union, implying that $\bigcap_{\alpha \in J} A_\alpha$ is closed.
(iii) Just like in (ii), consider the closed sets $A_1, A_2, \cdots A_n$. Then, we have

$$X - (\bigcup_{1 \le i \le n} A_i) = \bigcap_{1 \le i \le n} (X - A_i)$$

Since the sets $X - A_i$ are open, so is their intersection, implying that $\bigcup_{1 \le i \le n} A_i$ is closed. ∎

Using the properties of closed sets, we come up with some new tools.

Definition 3.3. Given a subset A of a topological space X, the interior of A, $IntA$, is defined as the union of all open sets contained in A. Moreover, the closure of A, $\bar{A}$, is defined as the intersection of all closed sets containing A.

From this definition, we see that $IntA$ is an open set and $\bar{A}$ is a closed set. Moreover, we have the chain of inclusions

$$IntA \subset A \subset \bar{A}$$

Notice that if A is open, $A = IntA$ and if A is closed, $A = \bar{A}$.

Theorem 3.4. *We have the following statements:*
(i) $x \in \bar{A}$ if and only if every open set U containing x intersects A.
(ii) Supposing X is generated by the basis $\mathcal{B}$, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A.

Proof. (i) We prove the contrapositive. If $x \notin \bar{A}$, then $U = X - \bar{A}$ is an open set containing x that does not intersect A. To prove the other contrapositive, if there is an open set containing x that does not intersect A, $X - U$ is a closed set containing x that intersects A. Since $X - U$ contains the closure, $\bar{A}$, we must have that $x \notin \bar{A}$, as desired.
(ii) This follows from (i) by using the fact that if every open set containing x intersects A, so does every basis element B containing x and that if every basis element containing x intersects A, then so does every open set U containing x (because U contains basis elements that contain x). ∎

We now introduce some special terminology.

Definition 3.5. We say that an open set U is a neighborhood of a point x if U is an open set containing x.

Definition 3.6. If A is a subset of the topological space X, we say that a point $x \in X$ is a limit point (or accumulation point) if every neighborhood of x intersects A at some point other than x itself. In other words, x is a limit point if it belongs to the closure of $A - \{x\}$.

The important thing about limit points is that they, together with the set A form its closure!

Theorem 3.7. *For $A \subseteq X$, let A' be the set of limit points of A. Then,*

$$\bar{A} = A \cup A'$$

Corollary 3.8. *$A \subseteq X$ is closed if and only if it contains all of its limit points.*

The notion of a limit point inspires us to think of convergence in terms of topology. Normally, a sequence $\{x_i\} \in \mathbb{R}$ converge to some limit x if for every $\epsilon > 0$, there is a positive integer N for which $|x_n - x| < \epsilon$ for all $n \geq N$. In the context of topology, we see a similar definition.

Definition 3.9. A sequence of points $x_1, x_2, \cdots$ in the topological space X is said to converge to a limit x if for each neighborhood U of x, there is some positive integer N for which $x_n \in U$ for all $n \geq N$.

In fact, it is evident that in an arbitrary topological space, it is possible for a sequence to converge to more than one point! To restrict ourselves to convergence to one point, we have the following definition:

Definition 3.10. A topological space is called Hausdorff if for each pair of points $x_1, x_2 \in X$, there exist disjoint neighborhoods U_1 and U_2 containing x_1 and x_2, respectively.

Theorem 3.11. *Every finite point set in a Hausdorff space is closed.*

Proof. It suffices to show every singleton $\{x_0\}$ is closed. If $x \in X$ is different from x_0, then x and x_0 have disjoint neighborhoods U and V. In other words, $\{x_0\}$ has no limit points. Thus, it is closed. ■

It turns out that this condition - every finite point set being closed - is weaker than the Hausdorff condition. If a set satisfies this condition, we say that it satisfies the T_1 axiom.

Proposition 3.12. *The T_1 axiom is equivalent to the following statement: for every $a, b \in X$ there is a neighborhood of a not containing b and a neighborhood of b not containing a.*

Proof. In one direction, if every finite point set is closed, then the sets $\{a\}$ and $\{b\}$ are closed. Thus, their complements are open. In particular, the complement of $\{a\}$ is a neighborhood of b not containing a and the complement of $\{b\}$ is a neighborhood of a not containing b, as desired.

In the other direction, if there is a neighborhood of a not containing b, then a is not a limit point of b. Since a and b are arbitrary, it follows that b has no limit points, and must be its own closure, i.e., $\{b\}$ is closed. Similarly, $\{a\}$ is closed as well, which concludes our proof. ■

Proposition 3.13. *Let X be a topological space satisfying the T_1 axiom. Then the point x is a limit point of a subset A of X if and only if every neighborhood of x contains infinitely many points of A.*

Proof. If every neighborhood of x contains infinitely many points of A, it is certainly a limit point of A. Conversely, let x be a limit point of A, and suppose some neighborhood U of x intersects A in finitely many points; let $\{x_1, x_2, \cdots x_m\}$ be the points of $U \cap (A - x)$. Then, $X - \{x_1, x_2, \cdots x_m\}$ is open (using the T_1 axiom) and

$$U \cap (X - \{x_1, x_2, \cdots x_m\})$$

is a neighborhood of x that does not intersect the set $A - \{x\}$ at all. This contradicts the fact that x is a limit point, meaning that every neighborhood of x must contain infinitely many points of A. ■

While this is powerful, the Hausdorff axiom is even more so.

Theorem 3.14. *If X is a Hausdorff space, then every sequence of points in X converges to at most one point.*

Proof. Suppose that x_n is a sequence of points of X converging to x. If $y \neq x$, then let U and V be disjoint neighborhoods of x and y, respectively. Since U contains x_n for all but finitely many n, the set V cannot. Thus, the sequence cannot converge to y. ■

One final statement we will make in this section ties back to the product topology:

Theorem 3.15. *The product of two Hausdorff spaces is a Hausdorff space.*

4. Continuous Functions and Homeomorphisms

At last, it is time to introduce functions in our world of topology!

Definition 4.1. Let X and Y be two topological spaces. A function $f : X \to Y$ is said to be continuous if for every open subset V of Y, $f^{-1}(V)$ is an open subset of X.

Often to prove continuity, it suffices to show that the inverse image of a basis element is open. Indeed, let $\mathcal{B}$ be the basis for Y. Then, an open set $V \subseteq Y$ can be written as a union of basis elements

$$V = \bigcup_{\alpha \in J} B_\alpha$$

which tells us that

$$f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha)$$

so that $f^{-1}(V)$ is open if $f^{-1}(B)$ is open, for each basis element $B \in \mathcal{B}$.

Theorem 4.2. *If X and Y are topological spaces and $f : X \to Y$ is a function, then the following statements are equivalent:*

(i) f is continuous

(ii) For every subset A of X, $f(\bar{A}) \subset f(\bar{A})$

(iii) For every closed set $B \subseteq Y$, $f^{-1}(B)$ is closed in X.

(iv) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood of U of x such that $f(U) \subset V$

While we omit the proof of this theorem (left as elementary exercises for the reader), if condition (iv) holds for the point x of X, we say that f is continuous at the point x.

Next, we come to the subject of homeomorphisms.

Definition 4.3. We say that a function f is a homeomorphism if both $f : X \to Y$ and $f^{-1} : Y \to X$ are continuous.

Essentially, a homeomorphism is a bijective correspondence between open sets. If U is any open set of X, then $f : X \to Y$ being a homeomorphism implies that $f(U)$ is open if and only if U is open. This is analogous to the notion of an isomorphism between groups, fields, or rings in algebra which preserves algebraic structure; homeomorphisms preserve topological structure.

Theorem 4.4. *If X, Y, Z are topological spaces, we have the following assertions:*
(i) If $f : X \to Y$ maps all of X to the point $y_0 \in Y$, then f is continuous
(ii) If A is a subspace of X, then the inclusion $j : A \to X$ is continuous.
(iii) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$
(iv) If $f : X \to Y$ is continuous, then the restriction map $f|A : A \to Y$ (for some subspace A of X) is continuous
(v) If X can be written as the union of open sets U_α such that $f|U_\alpha$ is continuous for each α, then f is continuous

We omit the proof of this theorem. These statements are fairly straightforward to prove using the definition of continuity, but also very useful in their own rights.

We end with one las theorem:

Theorem 4.5 (Maps into Products)**.** *Let $f : A \to X \times Y$ be given by the equation*

$$f(a) = (f_1(a), f_2(a))$$

Then, f is continuous if and only if $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous.

Proof. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the two projection maps to each factor; it is easy to show both are continuous. Then, we have

$$f_1(a) = \pi_1(f(a)) \text{ and } f_2(a) = \pi_2(f(a))$$

Thus, if f is continuous, then f_1 and f_2 are continuous by composition. Conversely, suppose f_1 and f_2 are continuous. We must show that for each basis element $U \times V$ in $X \times Y$, $f^{-1}(U \times V)$ is open. A point a is in $f^{-1}(U \times V)$ if and only if $f(a) \in U \times V$, i.e., $f_1(a) \in U$ and $f_2(a) \in V$. Thus,

$$f^{-1}(U \times V) = f^{-1}(U) \cap f^{-1}(V)$$

whence we have that $f^{-1}(U \times V)$ is open (intersection of open sets), as desired. ■

5. The Quotient Topology

Just like the product topology, we have the quotient topology. Although it sounds awkward, the quotient topology is rather easy to motivate; for example, a torus can be constructed by morphing a rectangular surface into a cylinder and stretching around the ends so that they meet. Formalizing this ”cut-and-paste” technique will make use of the quotient topology.

Definition 5.1. Let X and Y be topological spaces and let $p : X \to Y$ be a surjective map. The map p is said to be a quotient map if a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

This condition is stronger than continuity and referred to as ”strong continuity.” An equivalent condition is for $A \subseteq Y$ to be closed if and only if $p^{-1}(A)$ is closed.

Definition 5.2. We call a map $f : X \to Y$ an open map if for each open set U in X, $f(U)$ is open in Y as well. Similarly, we call it a closed map if for each closed set A in X, $f(A)$ is closed in Y as well.

From this definition, it follows that if p is a surjective continuous map that is either open or closed, it is a quotient map.

At last, we shall construct our topology.

Definition 5.3. If X is a space and A is a set, and if $p : X \to A$ is a surjective map, there exists exactly one topology τ on A relative to which p is a quotient map; this is known as the quotient topology (induced by p).

This topology consists of subsets U of A for which $p^{-1}(U)$ is open in X. To verify that it is indeed a topology, first notice that $\emptyset$ and A are in the topology (i.e. are open) because their inverse images are $\emptyset$ and X, respectively. The other two conditions follow quickly from

$$p^{-1}(\bigcup_{\alpha\in J} U_\alpha) = \bigcup_{\alpha\in J} p^{-1}(U_\alpha)$$

and

$$p^{-1}(\bigcap_{1\leq i\leq n} U_i) = \bigcap_{1\leq i\leq n} p^{-1}(U_i)$$

One of the reasons for having the quotient topology is the following:

Definition 5.4. Let X be a topological space and $X*$ be a partition of X into disjoint subsets whose union is X. Let $p : X \to X*$ be the surjective map carrying each element of X to the partition representative containing it. In the quotient topology induced by p, $X*$ is called a quotient space of X.

Observe that there is an equivalence relation on X for which elements of $X*$ are equivalence classes. Thus, the topology of $X*$ can be seen in the following manner: a subset U of $X*$ is a collection of equivalence classes for which $p^{-1}(U)$ is the union of those equivalence classes. In other words, an open set of $X*$ is a collection of equivalence classes whose union is an open set in X!

Using this topology, we can formalize the transformation of a rectangle into a torus.

6. Connectedness and Compactness

Now, we come to a brief section that will explore the notion of connectedness and compactness. These help define the notion of uniform continuity and existence theorems in analysis.

Definition 6.1. Let X be a topological space, as usual. Then, a separation of X is a pair (U, V) of disjoint nonempty open subsets whose union is X. Moreover, X is said to be connected if it has no separations.

Another way to think about connectedness is that the only subsets of X that are both open and closed are the empty set and X itself! Otherwise, if some nontrivial subset A were both open and closed, $X - A$ would also be open and closed and disjoint from A; their union would be X, contradicting connectedness.

Here is a quick and simple theorem that shows the usefulness of this definition:

Proposition 6.2. *If the sets C and D form a separation of X and Y is a connected subspace of X, then Y lies entirely within C or within D.*

Proof. Since C and D are open, so are $C \cap Y$ and $D \cap Y$; their union is Y. In order for this to not be a separation, we must have that one of the two intersections must be empty. In particular, this means Y is contained entirely in C or in D. ∎

Theorem 6.3. *The image of a connected space under a continuous map is connected.*

Proof. Suppose our continuous map is $f : X \to Y$. Since the restriction map is continuous, we may let $g : X \to f(X)$ be the surjective and continuous map that takes X to its image. Now, if $f(X)$ had a separation into two open sets A and B, then $g^{-1}(A)$ and $g^{-1}(B)$ - which exist by surjectivity - are two open sets in X whose union is X, a contradiction. We conclude that the image is connected. ∎

Now, we come to the harder but perhaps more useful idea: compactness

Definition 6.4. A collection $\mathcal{A}$ of subsets of X is said to be a covering of X if their union is X. $\mathcal{A}$ is said to be an open covering if each of these subsets are open.

Definition 6.5. A space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Although less intuitive, we state a few theorems that reveal its usefulness:

Theorem 6.6. *Every closed subspace of a compact space is compact.*

While we do not prove this theorem (left as a straightforward exercise for the reader), its proof is similar to that of the following theorem:

Theorem 6.7. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let Y be a compact subspace of the Hausdorff space X; we aim to show that $X - Y$ is open, so that it will follow that Y is closed.

Now, let x_0 be a point in $X - Y$. We claim that x_0 is outside of Y. Indeed, for any $y \in Y$, we may choose disjoint neighborhoods U_y and V_y of the points x_0 and y, respectively (by the Hausdorff condition). The collection $\{V_y \ : \ y \in Y\}$ is an open covering of Y, so by compactness, there is a finite subcollection

$$\{V_{y_1}, V_{y_2}, \cdots V_{y_n}\}$$

that covers Y. Clearly, the set

$$\bigcup_{1 \leq i \leq n} V_{y_i}$$

covers Y and is disjoint from the set

$$\bigcap_{1 \leq i \leq n} U_{y_i}$$

which contains x_0. Thus, U is a neighborhood of x_0 disjoint from Y, indicating that $X - Y$ is open, as desired. ∎

Theorem 6.8. *The image of a compact space under a continuous map is compact.*

Proof. Let $f : X \to Y$ be continuous with X compact. Let $\mathcal{A}$ be a covering of $f(X)$ by sets open in Y. Then, the set

$$\{f^{-1}(A) \ : \ A \in \mathcal{A}\}$$

is an open covering of X; in particular, there is a finite subcollection $f^{-1}(A_1), f^{-1}(A_2), \cdots f^{-1}(A_n)$ that covers X. Then, $A_1, A_2, \cdots A_n$ is a finite open covering of $f(X)$, as desired. ■

The idea of compactness when dealing with continuous maps will come up in the later sections. We will introduce one last theorem (without proof) that conveys a nice property of compact sets.

Theorem 6.9. *The product of finitely many compact spaces is compact.*

7. Topological Groups

We now introduce the main topological notions in this paper.

Definition 7.1. A topological group is a group G equipped with a topology such that the map $G \times G \to G$ that sends $(g, h) \mapsto gh$ (for $g, h \in G$) is continuous and the map $G \to G$ sending $g \mapsto g^{-1}$ (for $g \in G$) is continuous.

Actually, the latter map defines a homeomorphism. Moreover, we see that translation by any element of G defines a homeomorphism. Thus, we see that the group is translation-invariant and that for any $g \in G$ and $U \subseteq G$ the following three statements are equivalent:
(i) U is open (or U^{-1} is open, because inversion is a homeomorphism)
(ii) gU is open
(iii) Ug is open
Next, let $Homeo(X)$ denote the set of all homeomorphisms from $X \to X$ for some topological space X. Then, let $S \subseteq Homeo(X)$.

Definition 7.2. We say that S (defined as above) is a homogeneous space if for all $x, y \in X$, there exists an $f \in S$ such that $f(x) = y$.

Clearly, the topological group G is homogeneous under itself because for any $g, h \in G$ the left-translation homeomorphism $x \mapsto gh^{-1}x$ sends g to h. From this, we can see that the entire topology is determined.

Definition 7.3. We say that $U \subseteq X$ of a topological space X is a neighborhood of some $x \in X$ if x lies in the interior of U, i.e., in the largest open subset contained in U.

Before we move on to our first proposition, we come to one last, purely group-theoretic definition.

Definition 7.4. A subset $S \subseteq G$ is called symmetric if $S = S^{-1}$.

Proposition 7.5. *Let G be a topological group. Then, the following assertions hold true:*
(i) Every neighborhood U of the identity contains a neighborhood V of the identity such that $VV \subseteq U$.
(ii) Every neighborhood U of the identity contains a symmetric neighborhood V of the identity.
(iii) If H is a subgroup of G, then so is its closure.
(iv) Every open subgroup of G is also closed.
(v) If K_1 and K_2 are compact subsets of G, then so is K_1K_2.

Proof. (i) Assume that U is continuous and consider the continuous map $\phi : U \times U \to G$. Clearly, $\phi^{-1}(U)$ is open and contains (e, e). From the topology on $U \times U$, there exist open sets V_1 and V_2 such that $(e, e) \in V_1 \times V_2$. Then, let $V = V_1 \cap V_2$. Then, $VV \subseteq U$.

(ii) Clearly, $g \in U \cap U^{-1} \iff g, g^{-1} \in U$, so $V = U \cap U^{-1}$ is our desired symmetric neighborhood.

(iii) It suffices to show that the closure property holds. If g and h are two points in the closure of $H \subseteq G$, then g and h are convergent nets of elements of H. In particular, the product gh is also a convergent net of elements of H. Thus, closure holds and so the closure of H is a subgroup of G as well.

(iv) Take some open subgroup $H \subseteq G$. Then, H is simply the complement of the union of its nontrivial cosets, i.e., nontrivial translates. If H is open, then these translates are also open. This implies that H is also closed since it is a complement of an open set.

(v) K_1K_2 is the image of $K_1 \times K_2$ under the map $(k_1, k_2) \mapsto k_1k_2$, which is compact. ■

Corollary 7.6. *Using (i) and (ii), we see that every neighborhood of the identity U contains a symmetric neighborhood V such that $VV \subseteq U$.*

8. Separation Properties and Quotient Spaces

We first explore some properties of topological groups. We will see that the T_1 condition is equivalent to the Hausdorff condition in topological groups.

Proposition 8.1. *Let G be a topological group. Then, the following are equivalent:*
(i) G is T_1
(ii) G is Hausdorff
(iii) Every point of G is closed

Proof. (i) ⇒ (ii) From Section 3, we know that if a space X satisfies the T_1 axiom, it satisfies the following: if $a, b \in X$, then each point has a neighborhood not containing the other. Using this fact tells us that for any $g, h \in G$ there is an open neighborhood U of the identity e lacking the point gh^{-1}. According to Corollary 6.6, U contains a symmetric neighborhood V such that $VV \subseteq U$. Then Vg and Vh are disjoint neighborhoods of g and h, respectively; otherwise, $gh^{-1} \in V^{-1}V = VV \subseteq U$, a contradiction. We conclude that G is Hausdorff.

(ii) ⇒ (iii) Every point in a Hausdorff (or even T_1) space is closed.

(iii) ⇒ (i) Since every point of G is closed, all the finite point sets are closed, implying that G is T_1, as desired. ■

If H is a subgroup of a topological group G, then the set of left cosets G/H acquires the quotient topology. This means we have the canonical map $p : g \to gH$ for all $g \in G$ (each element maps to its equivalence class). Thus, U is open in G/H if $p^{-1}(U)$ is open in G. In fact, we can go even deeper: when H is a normal subgroup, G/H is a group, which we will prove is a topological group!

Now, we get to the main quotient construction theorem.

Theorem 8.2. *Let G be a topological group and H be a subgroup of G. Then, we have the following:*
(i) The quotient space G/H is homogeneous under G
(ii) The canonical projection $p : G \to G/H$ is an open map
(iii) The quotient space G/H is T_1 if and only if H is closed

(iv) The quotient space G/H is discrete if and only if H is open. Moreover, if G is compact, then H is open if and only if G/H is finite

(v) If H is a normal subgroup, then G/H is a topological group with respect to the quotient operation and the quotient topology

(vi) Let H be the closure of $\{e\}$ in G. Then H is normal and the quotient G/H is Hausdorff with respect to the quotient topology.

Proof. (i) An element $x \in G$ acts on G/H by left translation: $gH \mapsto xgH$. The inverse map takes the same form, so left translation is a homeomorphism of G/H. Thus, it suffices to show that left translation is an open mapping. If U' is an open subset of G/H, then by the quotient topology, its inverse image is the open set U of G. Then, the inverse image of gU' is gU, an open subset of G. Thus, gU' is open, and left translation is an open map, as desired.

(ii) Let V be an open set in G; we must show that $p(V)$ is open in G/H. By definition of the quotient map, $p(V)$ is open if and only if $p^{-1}(p(V)) = V \times H$ is open in G. Since V is open, so are any of its right translates; in particular, for any $h \in H$, Vh is open. Thus, by taking unions of these translates, we see that

$$V \times H = \bigcup_{h \in H} Vh$$

is open as well.

(iii) The quotient space G/H is closed if and only if each finite point set in G/H is closed; in particular, G/H is closed if and only if each singleton is closed. Since a coset of H is its own inverse image under the quotient map, the coset is closed if and only if it is closed in G. But by homogeneity, each coset of H is closed in G if and only if H itself is closed in G! Thus, (iii) follows.

(iv) Using (ii), we see that H is open if and only if H is open in G/H. Since G/H is homogeneous under G, it follows that gH is open in G/H for every $g \in G$. Because all the singletons $\{gH\}$ are open, it follows that H is open in G/H if and only if the topology on G/H is discrete.

Now, if G is compact, then so is G/H by continuity of the map p. Then, H is open if and only if G/H is discrete, whence we obtain that H is open if and only if G/H is both compact and discrete. The latter is just another way of saying that G/H is finite, so we are done.

(v) Let T_g denote left translation by g, so that $T_g(x) = gx$ and I and I' denote the group inverse maps in G and G/H, respectively (notice that G/H is a group since H is normal). Moreover, we also have the canonical projection $p : G \to G/H$. For any $x \in G$, we have

$$(p \circ T_g)(x) = gxH = (gH)(xH) = (T_{p(g)} \circ p)(x)$$

and

$$(p \circ I)(x) = x^{-1}H = (I' \circ p)(x)$$

In each of these equations, the maps commute. Since p is an open map by (ii) and T_g and I are continuous, it follows that $T_{p(g)}$ and I' are continuous as well, making G/H a topological group.

(vi) Since $\{e\}$ is a subgroup of G, so is its closure H. Moreover, we claim that H is normal. The conjugation $x \mapsto g^{-1}xg$ for any $g \in G$ is a group automorphism and also a homeomorphism (it is the composition of two translations). Thus, if H is a closed subgroup, so is $g^{-1}Hg$. Similarly, gHg^{-1} is closed as well. Now, since H is the closure of $\{e\}$, it is contained in every closed subgroup of G; in particular, $H \subseteq g^{-1}Hg$. Also, $H \subseteq gHg^{-1}$, which means $g^{-1}Hg \subseteq g^{-1}(gHg^{-1})g = H$. Thus, H is its own conjugate and is normal. Then, (v) tells us that G/H is a topological group with respect to the quotient topology. Finally, (iii) tells us that G/H is T_1, and thus Hausdorff as well. ∎

What's really interesting about part (vi) is that it reveals every topological group projects (via homomorphism) onto a topological group that is Hausdorff! Since every topological group is related to another Hausdorff topological group, let us investigate the Hausdorff property a little further:

Proposition 8.3. *Let G be a Hausdorff topological group. Then the following two properties hold:*

(i) The product of a closed subset F and a compact subset K is closed.

(ii) If H is a compact subgroup of G, then $p : G \to G/H$ is a closed map.

Proof. (i) We do this by showing $FK = \bar{FK}$. Let z be a point belonging to $\bar{FK}$. Then, there are two sequences $x_\alpha \in F$ and $y_\alpha \in K$ such that $\{x_\alpha y_\alpha\}$ converges to z. Since compact sets in Hausdorff spaces are sequentially compact, we can replace our sequence with a subsequence $\{y_\alpha\}$ that converges to $y \in K$. We claim that this forces x_α to converge to $zy^{-1} \in F$, showing that z lies in FK, which would mean that FK is closed. To prove this claim, we consider an arbitrary open neighborhood U of the identity. Then, there exists a symmetric neighborhood $V \subset U$ such that $VV \subseteq U$. From this, we see that the nets $\{z^{-1}x_\alpha y_\alpha\}$ and $\{y_\alpha^{-1}y\}$ are both eventually in V, meaning that

$$z^{-1}x_\alpha y_\alpha y_\alpha^{-1} y = z^{-1}x_\alpha y$$

. Thus, x_α converges to zy^{-1}, as desired.

(ii) Let X be any closed subset of G. Then, under p, X maps to XH, which by (i), is closed. To show the reverse is easy: if XH is closed, then so is X by the continuity of the quotient map. ∎

9. Some Parting Remarks

The properties of topological groups prove to be invaluable in analysis. However, less obvious is the fact that they are quite useful in algebraic number theory. The study of topological groups can further be refined through measure theory and representation theory. Moreover, the theorems of local class field theory can be formulated based on topology! In fact, Tate's thesis - fourier analysis on number fields - takes its roots in the building blocks of analysis. We hope the reader has learned some useful concepts from this paper and inclined to learn more about related topics!

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