FOURIER-ANALYTIC TECHNIQUES

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1. Introduction

In this paper, we explain how to use Fourier analysis to prove that certain measure-preserving transformations are ergodic. We first define Fourier series for a function f:

Definition 1.1. A Fourier series of f is $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi inx}$, where

$$a_n = \int_{\mathbb{R}/\mathbb{Z}} f(x)e^{2\pi inx}dx$$

.

One important property of Fourier series is that two Fourier series represent the same function almost everywhere if and only if their Fourier coefficients agree. This will allow us to compare the Fourier coefficients of a function and a function compose a transformation to tell whether function is constant almost everywhere.

The Riemann-Lebesgue Lemma, which tell us that the Fourier coefficients approach 0 at $\pm\infty$, is useful for Fourier analysis:

Lemma 1.2. *Let f be an integrable function with respect to the Lebesgue measure, and let a_n be the nth coefficient of the Fourier series of f. Then* $\lim_{|n|\to\infty} a_n = 0$.

Proof. To prove that the coefficients of the Fourier series approaches 0, we prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}/\mathbb{Z}} f(x)e^{2\pi inx}d\lambda = 0$$

. Let $A_1, A_2, A_3, \cdots$ be intervals, where $A_i = (a_i, b_i)$. Then, let $s(x) = \sum_{i=1}^{N} c_i \mathbb{1}_{A_i}$, where N is a positive integer and the c_i's are real coefficients. Then:

$$\lim_{n\to\infty}\int_{\mathbb{R}/\mathbb{Z}} s(x)e^{2\pi inx}d\lambda$$

$$= \lim_{n\to\infty}\int_{\mathbb{R}/\mathbb{Z}} (\sum_{i=1}^{N} c_i \mathbb{1}_{A_i})e^{2\pi inx}d\lambda$$

$$= \lim_{n\to\infty}\sum_{i=1}^{N}\int_{\mathbb{R}/\mathbb{Z}} c_i \mathbb{1}_{A_i}e^{2\pi inx}d\lambda$$

$$= \lim_{n\to\infty}\sum_{i=1}^{N} c_i \int_{a_i}^{b_i} e^{2\pi inx}dx$$

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$$= \sum_{i=1}^{N} \lim_{n\to\infty} c_i(\frac{1}{2\pi})(\frac{e^{2\pi i n b_i} - e^{2\pi i n a_i}}{in})$$

$$= 0$$

as desired.

Since measurable functions can be arbitrarily well approximated by simple functions, the proof extends to measurable functions as well. ☐

2. The Stone-Weierstrass Theorem

Theorem 2.1. *Let $[a,b]$ be an interval with real numbers a, b, and let $f : [a,b] \to \mathbb{R}$ be continuous. Then for every $\varepsilon > 0$, there exists a polynomial p such that for all $x \in [a,b]$, we have $|f(x) - p(x)| < \varepsilon$.*

Proof. We prove the theorem for $a = 0, b = 1$; to prove the general version, we can scale the functions so that the theorem holds for any interval with real endpoints.

The nth Bernstein polynomial of a continuous function on $[0,1]$ is defined as:

$$B_n(x,f) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k}$$

These polynomials can be used to approximate a continuous function on this interval. We can prove some basic properties of Bernstein polynomials:

Lemma 2.2. *Let f be a continuous function. Then, the Bernstein polynomials of f satisfy the following properties:*

(1) $B_n(x, cf) = cB_n(x,f)$
(2) $g(x) \leq f(x)$ *for all* x *implies* $B_n(x,g) \leq B_n(x,f)$
(3) $B_n(x, f + c) = B_n(x,f) + c$

Proof. (1) We have

$$B_n(x,cf) = \sum_{k=0}^{n} cf(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k}$$

$$= c\sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} = cB_n(x,f)$$

(2) Assume $g(x) \leq f(x)$ for all x. Then,

$$B_n(x,g) = \sum_{k=0}^{n} g(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k}$$

Since $g(\frac{k}{n}) \leq f(\frac{k}{n})$:

$$\leq \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} = B_n(x,f)$$

(3) By the definition of Bernstein polynomials, we have

$$B_n(x, f - f(c)) = \sum_{k=0}^{n}(f - f(z))(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k}$$

$$= \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k} - \sum_{k=0}^{n} f(z) \binom{n}{k} x^k (1-x)^{n-k}$$
$$= B_n(x,f) - f(z) \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$
$$= B_n(x,f) - f(z)(x+(1-x))^n$$
$$= B_n(x,f) - f(z)$$

as desired. □

Let $x, z \in [0,1]$, and let $\epsilon > 0$. Recall that $||f||_\infty = \operatorname{ess\,sup} f = \sup a : (f^{-1}(a,\infty)) = 0$. Let $M = ||f||_\infty$.

If $|x-z| \geq \delta$, then $|f(x) - f(z)| \leq 2M$ by the definition of M. We know that $\frac{|x-z|}{\delta} \geq 1$, so $|f(x)-f(z)| \leq 2M((\frac{x-z}{\delta})^2) + \frac{\epsilon}{2}$. On the other hand, since f is continuous, we know that there exists δ such that if $|x-z| \leq \delta$, then $|f(x)-f(z)| \leq \frac{\epsilon}{2} \leq 2M((\frac{x-z}{\delta})^2) + \frac{\epsilon}{2}$.

Then, by Lemma 2.2, we have

$$B_n(x,f) - f(z) = B_n(x, f - f(z))$$
$$\leq B_n(x, 2M((\frac{x-z}{\delta})^2) + \frac{\epsilon}{2}$$
$$= \frac{2M}{\delta^2} B_n(x,(x-z)^2) + \frac{\epsilon}{2}$$

We can evaluate $B_n(x,(x-z)^2)$:

Lemma 2.3. $B_n(x,(x-z)^2) = (x-z)^2 + \frac{1}{n}(x-x^2)$.

Proof. By the definition of Bernstein polynomials, we have:

$$B_n(x,(x-z)^2 = \sum_{k=0}^{n} (\frac{k}{n} - z)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=0}^{n} (\frac{k}{n})^2 \binom{n}{k} x^k (1-x)^{n-k} - \sum_{k=0}^{n} (\frac{2zk}{n}) \binom{n}{k} x^k (1-x)^{n-k} + \sum_{k=0}^{n} z^2 \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=0}^{n} (\frac{k}{n}) \binom{n-1}{k-1} x^k (1-x)^{n-k} - 2z \sum_{k=0}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} + z^2 \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$

The first two sums can start at 1 since the 0 terms evaluate to 0:

$$= \sum_{k=1}^{n} (\frac{k}{n}) \binom{n-1}{k-1} x^k (1-x)^{n-k} - 2zx \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k} + z^2 (x+(1-x))^n$$

Next, we shift the indices of the first two sums down:

$$= \sum_{k=0}^{n-1} (\frac{k+1}{n}) \binom{n-1}{k} x^{k+1} (1-x)^{n-k-1} - 2z \sum_{k=0}^{n} \binom{n-1}{k} x^k (1-x)^{n-k-1} + z^2$$

$$= \sum_{k=0}^{n-1} (\frac{k}{n}) \binom{n-1}{k} x^{k+1} (1-x)^{n-k-1} + \sum_{k=0}^{n-1} (\frac{1}{n}) \binom{n-1}{k} x^{k+1} (1-x)^{n-k-1} - 2zx + z^2$$

We can do the same for the first sum:

$$= \sum_{k=1}^{n-1} (\frac{k}{n}) \binom{n-1}{k} x^{k+1}(1-x)^{n-k-1} + x \sum_{k=0}^{n-1} (\frac{1}{n}) \binom{n-1}{k} x^k (1-x)^{n-k-1} - 2zx + z^2$$

$$= \sum_{k=0}^{n-1} (\frac{k+1}{n}) \binom{n-1}{k-1} x^{k+2}(1-x)^{n-k-2} + \frac{1}{n}(x) - 2zx + z^2$$

$$= \sum_{k=0}^{n-1} (\frac{k+1}{n}) \binom{n-1}{k-1} x^{k+2}(1-x)^{n-k-2} + \frac{1}{n}(x) - 2zx + z^2$$

Using the same methods and manipulation, the above is equal to $x^2 + \frac{1}{n}(x-x^2) - 2zx + z^2 = (x-z)^2 + \frac{1}{n}(x-x^2)$. □

By Lemma 2.3, we have

$$B_n(x, f) - f(z) \leq \frac{2M}{\delta^2} B_n(x, (x-z)^2) + \frac{\epsilon}{2}$$
$$= \frac{2M}{\delta^2}((x-z)^2 + \frac{1}{n}(x-x^2)) + \frac{\epsilon}{2}$$

If we plug in $x = z$, then

$$B_n(z, f) - f(z) \leq \frac{2M}{\delta^2}(\frac{1}{n}(x-x^2)) + \frac{\epsilon}{2}$$

Since $z - z^2 \leq \frac{1}{4}$ for $z \in [0, 1]$,

$$B_n(z, f) - f(z) \leq \frac{M}{2n\delta^2} + \frac{\epsilon}{2}$$

.

n can be arbitrarily large, so we have successfully approximated $f(z)$ for the interval $[0, 1]$. □

Not all measurable functions have Fourier expansions, but all continuous functions on $\mathbb{R}/\mathbb{Z}$ can be approximated by trigonometric polynomials, and all square-integrable functions (functions that are L^2 integrable) can be arbitrarily well approximated by finite trigonometric polynomials, which is a corollary of this theorem. This happens when we send x to $e^{2\pi inx}$:

Corollary 2.4. *Let f be an L^2-integrable function. Then, f can be arbitrarily well approximated by a Fourier series expansion.*

3. Examples

One example of how Fourier series can be used is for the rotation transformation R_α on $\mathbb{R}/\mathbb{Z}$.

Example. Let $X = \mathbb{R}/\mathbb{Z}$ and R_α be the rotation defined by $R_\alpha(x) = x + \alpha$, where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Let f be a measurable and L^2 integrable function. Then, if the Fourier series of f is

$$f(x) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi inx}$$

Then the Fourier series of $f \circ R_\alpha$ is:

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi in(x+\alpha)}$$

$$= \sum_{n\in\mathbb{Z}} (a_n e^{2\pi in\alpha}) e^{2\pi inx}$$

Comparing the coefficients, $a_n e^{2\pi in\alpha} = a_n$. Since $e^{2\pi in\alpha}$ cannot be 1 unless $n = 0$, this means that $a_n = 0$ for $n \in \mathbb{Z}, n \neq 0$, which means that f is constant almost everywhere. This implies that f is ergodic.

Another example is the doubling map, which uses the Riemann Lebesgue Lemma. Note that this can be generalized to any transformation $T(x) = \beta x - \lfloor \beta x \rfloor$:

Example. Let $X = \mathbb{R}/\mathbb{Z}$ and let $T(x) = 2x \pmod 1$. Suppose f is a measurable and L^2 integrable function. Then, if the Fourier series of f is:

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi inx}$$

Then the Fourier series of $f \circ T^j$ for some nonnegative integer j is:

$$f(x) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi in(x)(2^j)}$$

$$= \textstyle\sum_{n\in\mathbb{Z}} (a_{n(2^j)}) e^{2\pi inx}$$

By comparing the coefficients, we get $a_n = a_{2^j n}$. These coefficients will approach 0 as j approaches infinity, which implies that f is constant almost everywhere. Thus, T is ergodic.

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