Applications of Fourier Analysis in Ergodic Systems

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Abstract

It is essential in mathematics to be able to approximate functions with other functional constructions that are easier to work with. It is also useful to define the constraints which a function must follow for it to be able to be approximated by such constructions. Such is what was accomplished by Karl Weierstraß and Marshall Stone in the Stone-Weierstraß Theorem, which proves that if X is a compact Hausdorff space, a function $f : X \to \mathbb{R}$ need only be continuous to be approximated arbitrarily well by polynomials. The Weierstraß Approximation Theorem asserts that such functions can be approximated well by algebraic and trigonometric polynomials, and is essential in the use of Fourier Analysis to prove the ergodicity of measure-preserving transformations. Equipped with the Stone-Weierstraß Theorem, the Fourier Series can simplify many problems on $\mathbb{R}/\mathbb{Z}$, and prove the ergodicity of other measure-preserving transformations.

1 Fourier Analysis

Definition 1. Fourier Series

The *Fourier Series*, developed by French mathematician Joseph Fourier, is a method to globally approximate periodic functions. Let a_n and b_n be coefficients, the values of which will be determined later. Then, the Fourier Series is given by the infinite sum of a configuration of sines and cosines, denoted:

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

The intuition behind the Fourier Series is that there exists some sum of trigonometric functions, appropriately vertically shifted by some constant a_0, which have varying frequencies. The infinite sum of such functions will then approximate another periodic function arbitrarily well. The coefficient of the x inside the trigonometric functions, which is n, varies as the infinite sum progresses, and thus produces the varying frequencies necessary to approximate other functions. However, the trigonometric functions with varying frequencies each need to be scaled and weighted accordingly to produce the desired function. Such is the motivation for the coefficients a_n and b_n. We now generalize the Fourier Series to a function with period $2T$.

Proposition 1. The Fourier Series of a function $F(x)$ which has period $2T$, or frequency f, is given by

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{T}\right) + b_n \sin\left(\frac{\pi n x}{T}\right)$$

or

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f x) + b_n \sin(2\pi n f x).$$

Proof. We begin the proof by performing a change of variables. Let the variable y be defined as $y = \frac{\pi x}{T}$. Then, let the function $h(y)$ be defined as:

$$h(y) = F(x).$$

Then, we have $h(y) = F(\frac{Tt}{\pi})$. It is easy to see, due to the change of variables, that the function h has period 2π. Then, it is possible to represent h through a Fourier Series, given by

$$h(y) = a_0 + \sum_{n=1}^{\infty} a_n \cos(ny) + b_n \sin(ny).$$

Reverting to the variable x by noting that $y = \frac{\pi x}{T}$, and using $h(y) = f(x)$, we have

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi n x}{T}) + b_n \sin(\frac{\pi n x}{T})$$

for the Fourier Series of a period-$2T$ function. This Fourier Series can also be written in terms of the frequency f instead of the period $2T$, in which case it becomes

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f x) + b_n \sin(2\pi n f x)$$

■

Proposition 2. The vertical shift term a_0 is given by

$$a_0 = \frac{1}{2T} \int_0^{2T} F(x) dx.$$

Proof. Let $F(x)$ be some function which has period 2π. To determine the constant vertical shift a_0, we begin by integrating both sides of the Fourier Series of $F(x)$. Then, we have

$$\begin{aligned} \int_0^{2\pi} F(x) dx &= \int_0^{2\pi} a_0 dx + \int_0^{2\pi} \sum_{n=1}^{\infty} \Big[a_n \cos(nx) + b_n \sin(nx) \Big] dx \\ &= 2\pi a_0 + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \cos(nx) dx + \sum_{n=1}^{\infty} a_n \int_0^{2\pi} \sin(nx) dx. \end{aligned}$$

Since for $\forall n \in \mathbb{Z}$ the expression $\int_0^{2\pi} \cos nx dx$ evaluates to 0, the second term in the above equation becomes zero. The integral $\int_0^{2\pi} \sin(nx) dx$ also evaluates to zero for all integer n, so the only nonzero term in the above equation is $2\pi a_0$. Then, we have

$$\int_0^{2\pi} F(x) dx = 2\pi a_0.$$

Rearranging terms, we have

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} F(x)dx.$$

However, this is only true for functions which have period 2π. To generalize the solution for a_0 to a function with an arbitrary period $2T$, we must make a change of variables. Let the variable y be defined as $y = \frac{\pi x}{T}$, and the function $h(y)$ be defined as $h(y) = f(x)$. Note that $h(y)$ has period 2π. Then, when x becomes integer multiples of T, y reaches integer multiples of π. Writing out the a_0 term of the Fourier Series of $h(y)$, we have

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} h(y)dy.$$

Reverting to the variable x and the function $F(x)$ and changing the limits and coefficients accordingly, we have

$$a_0 = \frac{1}{2T}\int_0^{2T} F(x)dx$$

for the generalized solution to a_0. ■

Proposition 3. The Fourier Coefficient of the nth cosine function in the infinite sum is given by

$$a_n = \frac{1}{T}\int_0^{2T} F(x)\cos\left(\frac{\pi n x}{T}\right)dx.$$

Proof. We derive the Fourier Coefficient of the nth cosine function in the Fourier Series of $F(x)$ by first multiplying both sides of the Fourier Series by $\cos(kx)$, where k is some positive integer. Then, we have

$$F(x)\cos(kx) = a_0\cos(kx) + \cos(kx)\sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx).$$

After integrating both sides, we see that $\int_0^{2\pi} F(x)\cos(kx)dx$ is equal to:

$$a_0\int_0^{2\pi}\cos(kx)dx + a_n\sum_{n=1}^{\infty}\int_0^{2\pi}\cos(nx)\cos(kx)dx + b_n\sum_{n=1}^{\infty}\int_0^{2\pi}\sin(nx)\cos(kx)dx.$$

Since $\int_0^{2\pi}\cos(kx)dx = 0$, the first term can be forgotten. Now, note that the integral of the product of cosines, $\int_0^{2\pi}\cos(nx)\cos(kx)dx$, is equal to 0 for all integer n and k such that $n \neq k$.[1] When $n = k$, the integral evaluates to π. On the other hand, the expression $\int_0^{2\pi}\sin(nx)\cos(kx)dx$ evaluates to zero for all integer n and k regardless of whether or not $n = k$.[2] Then, the last nonzero term remaining is $a_k\pi$. After forgetting the zero terms, we see that

$$\int_0^{2\pi} F(x)\cos(kx)dx = a_k\pi.$$

[1] The evaluation of this integral is accomplished by using the product-to-sum formulas, one of which shows that $\int_0^{2\pi}\cos(nx)\cos(kx)dx = 0$ can be simplified to $\frac{1}{2}\int_0^{2\pi}\cos(nx+kx)+\cos(nx-kx)dx$, which also evaluates to zero.

[2] Again, the use of one of the product-to-sum formulas facilitates the evaluation of this integral. The integral simplifies to $\frac{1}{2}\int_0^{2\pi}\sin(nx+kx)+\sin(nx-kx)dx$, which evaluates to zero for all $n, k \in \mathbb{Z}^+$.

After exchanging n for k in both $\cos(kx)$ and a_k and rearranging terms, we have

$$a_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\cos(nx)dx.$$

Steps similar to those taken during the generalization of the a_0 term are used to generalized a_n to a function with arbitrary period $2T$. After the implementation of the change of variables given by $y = \frac{\pi x}{T}$ and $h(y) = F(x)$, the Fourier Series of $h(y)$ is

$$h(y) = a_0 + \sum_{n=1}^{\infty} a_n \cos(ny) + b_n \sin(ny),$$

and the a_n term is given by

$$a_n = \frac{1}{\pi}\int_0^{2\pi} h(y)\cos(ny)dy.$$

We now revert to old variables, use the function F instead of h, and change limits accordingly. Then, we have:

$$a_n = \frac{1}{T}\int_0^{2T} F(x)\cos(\frac{\pi n x}{T})dx,$$

thus ending the derivation of the generalized solution to a_n. ■

Proposition 4. The Fourier Coefficient of the nth sine function in the infinite sum is given by

$$b_n = \frac{1}{T}\int_0^{2T} F(x)\sin\left(\frac{\pi n x}{T}\right)dx.$$

Note that b_0 is defined to be zero, so the above equation works for all nonzero positive integer values of n.

Proof. An approach similar to that taken in the derivation of a_n is used to derive b_n. We begin by first multiplying both sides of the Fourier Series by $\sin(kx)$, where k is some positive integer. Then,

$$F(x)\sin(kx) = a_0\sin(kx) + \sin(kx)\sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx).$$

Integrals are taken on both sides of the equation, to yield:

$$\int_0^{2\pi} F(x)\sin(kx)dx = a_0\int_0^{2\pi}\sin(kx) + \sum_{n=1}^{\infty} a_n\int_0^{2\pi}\cos(nx)\sin(kx)$$

$$+\sum_{n=1}^{\infty} b_n\int_0^{2\pi}\sin(nx)\sin(kx)dx.$$

The integrals $a_0\int_0^{2\pi}\sin(kx)$ and $\sum_{n=1}^{\infty} a_n\int_0^{2\pi}\cos(nx)\sin(kx)$ both evaluate to zero, so they can be forgotten. The integral $\sum_{n=1}^{\infty} b_n\int_0^{2\pi}\sin(nx)\sin(kx)dx$ also evaluates to zero, but takes a nonzero value when $n = k$, which is π. Thus,

$$\int_0^{2\pi} F(x)\sin(kx)dx = \pi b_k.$$

Rearranging terms and exchanging n for k, we have:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\sin(nx)dx.$$

Since this holds true only for functions with period 2π, it is necessary to generalize the formula to fit a function with arbitrary degree $2T$. An approach similar to those taken to generalize a_0 and a_n will be used here. Recall the change of variables $y = \frac{\pi x}{T}$ and the definition of the function $h(y) = F(x)$. The Fourier Series of $h(y)$ is given by

$$h(y) = a_0 + \sum_{n=1}^{\infty} a_n\cos(ny) + b_n\sin(ny),$$

and the b_n term is calculated using the fact that

$$b_n = \frac{1}{\pi}\int_0^{2\pi} h(y)\sin(ny)dy$$

Conducting the reversion to the variable x, writing the expression in terms of $F(x)$, and changing the limits and coefficients accordingly, we have

$$b_n = \frac{1}{T}\int_0^{2T} F(x)\sin(\frac{\pi nx}{T})dx$$

for the generalized solution to the Fourier Coefficient b_n. ■

Definition 2. The Complex Fourier Series

The *Complex Fourier Series* is simply another representation of the regular Fourier Series. However, the Complex Fourier Series is much easier to work with, for it expresses the series using exponents and a single coefficient-generating function instead of trigonometric functions and multiple coefficients. Let f be the frequency of an arbitrary $F(x)$. Note that if $2T$ is the period of F, then $f = \frac{1}{2T}$. The Complex Fourier Series of a function $F(x)$ with frequency f is given by:

$$F(x) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi inf x}.$$

The Fourier Coefficient function C_n will be defined in the derivation below.

Derivation. Before giving the derivation of the Complex Fourier Series, we will need to establish a few preliminaries.

(i) Euler's formula is given by $e^{i\phi} = \cos(\phi) + i\sin(\phi)$. Adding $e^{-i\phi}$ to both sides of the equation, we have

$$\cos(\phi) = \frac{e^{i\phi} + e^{-i\phi}}{2}.$$

(ii) Similarly, it is possible to isolate $\sin(\phi)$ from Euler's formula. Subtracting $e^{-i\phi}$ from both sides of the equation, we have

$$\sin(\phi) = \frac{e^{i\phi} - e^{-i\phi}}{2i}.$$

Now that the preliminary material has been established, we can move on with the derivation of the Complex Fourier Series. Proposition 1 shows that the Fourier Series of a function with frequency f is given by

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n f x) + b_n \sin(2\pi n f x)$$

Here, we make the substitutions $\cos(\phi) = \frac{e^{i\phi}+e^{-i\phi}}{2}$ and $\sin(\phi) = \frac{e^{i\phi}-e^{-i\phi}}{2i}$ for $\phi = 2\pi n f x$. Then, we have

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left[\frac{a_n(e^{2\pi i n f x} + e^{-2\pi i n f x})}{2} + \frac{b_n(ie^{-2\pi i n f x} - ie^{2\pi i n f x})}{2} \right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - ib_n}{2}\right) e^{2\pi i n f x} \right] + \sum_{n=1}^{\infty} \left[\left(\frac{a_n + ib_n}{2}\right) e^{-2\pi i n f x} \right]$$

Now, all coefficients are combined into a single function C_n. Then, the Complex Fourier Series of $F(x)$ is given by

$$F(x) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi i n f x}.$$

We define the Fourier Coefficient function C_n as

$$C_n = \left\{ \begin{array}{ll} \frac{1}{2}(a_{|n|} + ib_{|n|}) & \text{for } n < 0 \\ \frac{1}{2}(a_0) & \text{for } n = 0 \\ \frac{1}{2}(a_n - ib_n) & \text{for } n > 0 \end{array} \right\} = \langle F(x) e^{-2\pi i n f x} \rangle = \frac{1}{2T} \int_0^{2T} F(x) e^{-2\pi i n f x} dx.$$

■

The Fourier Series is extremely useful in Ergodic Theory, since it can be used in conjunction with certain theorems to prove that measure-preserving transformations are ergodic. The Fourier Series is similar to the *Taylor Series*, which approximates highly differentiable functions arbitrarily well over closed intervals (locally). However, the Fourier Series is more versatile than the Taylor Series when working with periodic functions, for it approximates functions globally, and does not require the functions it approximates to be highly differentiable. It is important to prove that certain functions can be approximated arbitrarily well by other functional constructions before using Fourier Analysis to approximate certain measure-preserving transformations in order to prove results in Ergodic Theory. A special result of the Stone-Weierstraß Theorem, the *Weierstraß Approximation Theorem*, accomplishes precisely this. However, we will need to establish some groundwork before discussing the Stone-Weierstraß Theorem and its useful results.

2 The Stone-Weierstraß Theorem

Definition 3. Algebra

An *algebra* $\mathcal{A}$ is a vector space in $\mathbb{R}$ equipped with a multiplication operation. More formally, an algebra is a vector space in $\mathbb{R}$ with an associative bilinear product, denoted $p : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

Definition 4. Sub-algebra

A *sub-algebra* is a subset of an algebra which, in turn, also satisfies the conditions required to be labelled an algebra under the same operations. An alternate definition of a sub-algebra also exists; a *sub-algebra* is a sub-ring $\mathcal{A} \subset C[x_1, x_2]$ if for $\forall f, g \in \mathcal{A} : fg \in \mathcal{A}$. A sub-algebra is *unital* if the aforementioned operation $p : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ has an identity element.

Definition 5. Separating Points

A sub-ring, or a set of functions F *separates points* if $\forall x, y : x \neq y : \exists f \in F$ such that $f(x) \neq f(y)$.

Definition 6. Convolution

Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary smooth function, and let $g : [a, b] \to \mathbb{R}$ be a continuous function over the interval $[a, b]$. A *convolution* of two functions f and g, denoted $f * g = h$, is given by

$$h(x) = \int_a^b f(x - z)g(z)dz.$$

The convolution of two such functions is commutative, and is also a smooth function.

Theorem 1. The Stone-Weierstraß Theorem

Let X be a compact metric space and $\mathcal{A} \subset C(X, \mathbb{R})$ be a unital sub-algebra that separates points of X. Then X is dense in $C(X, \mathbb{R})$. This is equivalent to the statement that if $\mathcal{A} \subset C(X, \mathbb{R})$ and $\mathcal{A}$ is a sub-algebra that contains a non-zero constant function, then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$ if and only if it separates points of compact set X.

Proof. It will be proved in the work that follows that smooth functions can approximate continuous functions arbitrarily well over closed intervals. This will be done by proving the existence of a smooth function f such that the convolution of f and g, the function being approximated, gets arbitrarily close to g. This is accomplished by flattening, or distributing the weight of the function f, more equally over the interval on which g is defined.

Let $g : [a, b] \to \mathbb{R}$ be a continuous function over the interval $[a, b]$; this is the function that will be approximated by smooth functions. Then let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function that has a value of 0 everywhere but in the interval $[a, b]$, over which the function is positive. Suppose that the average of the function's "weight" over the interval $[a, b]$ is 1; in other words,

$$\int_a^b f(x)dx = 1.$$

It is possible to state the function f more specifically; let $k(x)$ be defined such that

$$k(x) = \left\{ \begin{array}{ll} e^{-2(x^2+1)}, & \text{for } x \in [-1, 1] \\ 0, & \text{else} \end{array} \right\}$$

It is true, then, that k is a smooth function, that k is non-negative over the interval $[-1, 1]$, and that $k = 0$ everywhere else. Thus, there exists an explicitly stated function that satisfies the criteria given for the function f. Next, let the function $f_n(x) = \frac{f(\frac{x}{n})}{n}$, and $\int_a^b f_n(x) = 1$ as well. Note that as n tends to ∞, the graph of f_n becomes more even, and the weight over the interval $[a, b]$ becomes more equally distributed. Note that the function g is uniformly continuous over the closed interval $[a, b]$. This, then, implies that for some $\varepsilon > 0$ and all $x, y \in [a, b]$, $\exists\delta > 0 : |f(x) - f(y)| < \varepsilon$, for $|x - y| < \delta$. Using the commutativity of $f_n * g$, we have

$$|(f_n * g) - g(x)| = \left| \int_a^b f_n(z)g(x-z)dz - g(x)\int_a^b f_n(z)dz \right|.$$

Then, by the triangle inequality, we have

$$\left| \int_a^b f_n(z)g(x-z)dz - g(x)\int_a^b f_n(z)dz \right| \leq \int_a^b |f_n(z)||g(x-z) - g(x)|dz.$$

Because f_n is nonzero only inside the interval $[\frac{a}{n}, \frac{b}{n}]$, the limits on the integral simplifies to $\frac{a}{n}$ and $\frac{b}{n}$. Then, we have

$$\int_a^b |f_n(z)||g(x-z) - g(x)|dz = \int_{\frac{a}{n}}^{\frac{b}{n}} |f_n(z)||g(x-z) - g(x)|dz$$

Note that for $\forall z \in [\frac{a}{n}, \frac{b}{n}]$, $|g(x-z) - g(x)| < \varepsilon$, given that n is sufficiently large, such that the inequality $0 < \frac{b-a}{n} < \delta$ holds. Then, for all values of $x \in [a, b]$, we have

$$|(f_n * g) - g(x)| \leq \int_{\frac{a}{n}}^{\frac{b}{n}} |f_n(z)||g(x-z) - g(x)|dz < \varepsilon \int_{\frac{a}{n}}^{\frac{b}{n}} |f_n(z)|dz < \int_a^b f_n(z)dz = 1.$$

Thus, the existence of a smooth function f such that its convolution with g approximates g arbitrarily well is proven. It follows from this result that all continuous functions can be approximated by smooth functions arbitrarily well. ■

Theorem 2. The Weierstraß Approximation Theorem

Let $C(\mathbb{R}, \mathbb{R})$ denote the space of all continuous real valued functions mapping from $\mathbb{R} \to \mathbb{R}$, and let $\mathbb{P}$ denote the set of all real-valued polynomials such that for every polynomial $p \in \mathbb{P}$, we have $p \in C(\mathbb{R}, \mathbb{R})$. Then, $\mathbb{P}$ is dense in $C(\mathbb{R}, \mathbb{R})$.

Proof. By the definition of a polynomial, $f, g \in \mathbb{P}$ implies that both $f + g$ and $f \cdot g$ are polynomials, and thus are in $\mathbb{P}$. Therefore, due to closure under the additive and multiplicative operations, $\mathbb{P}$ is a sub-algebra of $C(\mathbb{R}, \mathbb{R})$. Moreover, $\mathbb{P}$ contains all constants, as constants are simply degree zero polynomials. Next, we must prove that $\mathbb{P}$ separates points. Let $a, b \in \mathbb{R}$. Consider the polynomial given by $f(x) = x - a$. Then, $f(a) = 0$, and $f(b) = b - a$. Thus, $f(a) \neq f(b)$, so $\mathbb{P}$ separates points. Then, by the Stone-Weierstraß Theorem, $\mathbb{P}$ is dense in $C(\mathbb{R}, \mathbb{R})$. ■

Theorem 3. The Approximation Theorem for Trigonometric Polynomials

The Weierstraß Approximation Theorem can also be extended to describe the density of trigonometric polynomials in the set of all real-valued functions having period $x_2 - x_1$ and continuous over $[x_1, x_2]$. Let $\tilde{C}[x_1, x_2]$ represent the aforementioned set. Then, trigonometric polynomials are dense in $\tilde{C}[0, 2\pi]$. Trigonometric polynomials can then approximate periodic functions arbitrarily well. In other words, if $f \in \tilde{C}[0, 2\pi]$, then for $\forall x \in [0, 2\pi]$, there exists some $\varepsilon > 0$ and a trigonometric polynomial T such that

$$|f(x) - T(x)| < \varepsilon$$

Proof. Since trigonometric polynomials are smooth, and since the Stone-Weierstraß Theorem proves that smooth functions have the ability to approximate continuous real-valued functions arbitrarily well, trigonometric polynomials can approximate all functions in $\tilde{C}[x_1, x_2]$ arbitrarily well. ■

This result of the Weierstraß Approximation Theorem is indispensable, for it allows the use of Fourier Series to approximate and prove the ergodicity of certain transformations.

3 Rotations and Other Transformations

Proposition 5. The circle rotation $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is λ-preserving for all $\alpha \in \mathbb{R}$.

Proof. Let the transformation $R_\alpha(x) = (x+\alpha) \bmod 1$. For simplicity, we will rename this rotational transformation as T. Then, we must prove that if $A = [a, b]$ such that $a, b \in [0, 1)$,

$$\lambda(T^{-1}(A)) = \lambda(A).$$

It is easy to see that $\lambda(A) = b - a$, and $\lambda(T^{-1}(A))$ follows similarly. Note that

$$T^{-1}(A) = T^{-1}\Big([a, b]\Big) = [a - \alpha, b - \alpha]$$

Then, we have

$$\lambda\Big(T^{-1}(A)\Big) = (b - \alpha) - (a - \alpha) = b - a.$$

Thus, both $\lambda(A)$ and $\lambda(T^{-1}(A))$ equal $b - a$. Thus, the rotation T preserves Lebesgue measure λ. ■

Proposition 6. The Lebesgue measure is R_α-invariant.

Proof. For simplicity, we will be using T to replace R_α. Recall the definition of T-invariant measures: λ is T-invariant if for all real-valued continuous functions f,

$$\int f \circ T d\lambda = \int f d\lambda.$$

Then, let f have the Fourier Series

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}.$$

Then, $f \circ T$ has the Fourier Series

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x} e^{2\pi i n \alpha}.$$

We now take the integrals of f and $f \circ T$ with respect to λ.

$$\int f d\lambda = \int \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x} d\lambda = \sum_{n=-\infty}^{\infty} c_n \int e^{2\pi i n x} d\lambda$$

Note that $\int e^{2\pi i n x} d\lambda = 0$ for all n except for $n = 0$, at which the integral evaluates to one. Using this, we have:

$$\int f d\lambda = c_0.$$

Now we shall begin integrating $f \circ T$.

$$\int f \circ T d\lambda = \int \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx} e^{2\pi in\alpha} d\lambda = \sum_{n=-\infty}^{\infty} c_n e^{2\pi in\alpha} \int e^{2\pi inx} d\lambda$$

Again, we use the fact that $\int e^{2\pi inx} d\lambda$ behaves like the point mass at zero. Then, we have

$$\int f \circ T d\lambda = c_0.$$

Both $\int f d\lambda$ and $\int f \circ T d\lambda$ equal c_0, and are thus equal to each other. Thus, λ is R_α-invariant. ■

Proposition 7. R_α is ergodic for all $\alpha \notin \mathbb{Q}$.

Proof. We prove the ergodicity of irrational rotations on $\mathbb{R}/\mathbb{Z}$ by proving that every R_α-invariant measurable function is constant everywhere. Let $\alpha \notin \mathbb{Q}$ and $f \in L^2$ be R_α-invariant. Then, let f have the Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx}$$

This implies that $f \circ R_\alpha$ has the Fourier Series

$$f \circ R_\alpha = \sum_{n=-\infty}^{\infty} c_n e^{2\pi in(x+\alpha)} = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx} e^{2\pi in\alpha}$$

Since we assume f to be R_α-invariant, $f \circ R_\alpha(x)$ must be equal to $f(x)$. Then, we have:

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi inx} = \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx} e^{2\pi in\alpha}$$

By comparing Fourier Coefficients, we have:

$$c_n = c_n e^{2\pi in\alpha}$$

There exist two possibilities: either $e^{2\pi in\alpha} = 1$, or $c_n = 0$. Since n only equals 0 in one instance and $\alpha \neq 0$ because $\alpha \notin \mathbb{Q}$, $e^{2\pi in\alpha} \neq 1$. So, c_n must equal 0 whenever $n \neq 0$. When $n = 0$, the first term, c_0, appears. Since all terms of the Fourier Series of f are 0 save for c_0, f is constant almost everywhere. Thus, R_α for all $\alpha \notin \mathbb{Q}$ is ergodic.

We now prove that rational rotations on $\mathbb{R}/\mathbb{Z}$ are not ergodic by providing an example of a R_α-invariant function which is not constant. Let $\alpha = \frac{a}{b}$, where $a, b \in \mathbb{Z}$. Then, let $f(x) = e^{2\pi ibx}$. Then,

$$f \circ R_\alpha(x) = e^{2\pi ib(x+\frac{a}{b})} = e^{2\pi ibx} e^{2\pi ia}$$

Since a is an integer, $e^{2\pi ia}$ evaluates to 1, and $f \circ R_\alpha(x) = f(x)$. Thus, f is R_α-invariant. However, f is not constant. For this reason, when $\alpha \in \mathbb{Q}$, R_α is not ergodic. ■

Proposition 8. T_α is not ergodic if $\alpha \in \mathbb{Q}$.

We now prove that rational rotations on $\mathbb{R}/\mathbb{Z}$ are not ergodic by providing an example of a R_α-invariant function which is not constant. Let $\alpha = \frac{a}{b}$, where $a, b \in \mathbb{Z}$. Then, let $f(x) = e^{2\pi ibx}$. Then,

$$f \circ R_\alpha(x) = e^{2\pi ib(x+\frac{a}{b})} = e^{2\pi ibx} e^{2\pi ia}$$

Since a is an integer, $e^{2\pi ia}$ evaluates to 1, and $f \circ R_\alpha(x) = f(x)$. Thus, f is R_α-invariant. However, f is not constant. For this reason, when $\alpha \in \mathbb{Q}$, R_α is not ergodic. ■

Proposition 9. The doubling transformation, given by $T(x) = 2x \pmod 1$, is ergodic.

Proof. Let the *doubling transformation* $T : X \to X$ be defined as

$$T(x) = 2x \pmod 1.$$

We wish to again use the fact that T is ergodic if and only if every T-invariant function is constant almost everywhere. Then we again look for f such that $f \circ T$ is equal to f. We can let $f(x)$ have the Fourier Series given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}.$$

Then, for any $p > 0$, we have that $f \circ T^p$ could be represented by

$$f \circ T^p = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n 2^p x}.$$

Then, by comparing the Fourier Coefficients of both $f \circ T$ and $f \circ T^p$, we have that $c_n = c_{2^p * n}$. The Riemann-Lebesgue Lemma states that $c_n \to 0$ as $|n| \to \infty$. This implies that for all $n \neq 0$, $c_n = 0$. If this is true, the function is equal to 0 everywhere except for c_0. So, the functions are constant almost everywhere. This implies that T is ergodic. ∎

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