ENTROPY

KARTHIK BALAKRISHNAN

1. What is entropy?

Entropy in mathematics came from the question in Claude Shannon's book "A Mathematical Theory of Communication" where he considered how much information was conveyed by each letter or symbol in a message. Thus the concept of entropy was formalized. Later this idea generalized by Kolomogrov and Sinai for dynamical systems. We partition a finite space X into $P = (p_1, p_2, ...p_n)$ pieces. We want to not only measure how much "information" we get from each of the subsets of X, but also how much information we get from each additional subset. Thus we define entropy of a partitioned space X with measure μ in the following way [Wal]:

$$H(p_1, p_2, ...p_n) = -\sum_{i=1}^{n} \mu(p_i) log(\mu(p_i))$$

The entropy function as the following property:

- $H(p_1, ..., p_k) \geq 0$ and is 0 if and only if one $p_i = 1$
- $H(p_1, ..., p_n, 0) = H(p_1, ..., p_n)$
- H is maximized when $\mu(p_1) = ... = \mu(p_k)$

Now we want to define entropy not just for a partitioned space, but rather for a measure preserving function T. We want to know how well T mixes the partitions to give us the most partition. Thus we define a new entropy of a measure preserving function:

$$h(T) = sup_{P \in X} H(P)$$

Now this theorem might seem utterly useless, as we would have to consider all the different partitions to find the supremum. However, it turns out that if the partition we choose generates the σ-algebra, then we can say that the entropy of that partition is the entropy of the function.

To show the theorem above, we must first introduce some key ideas

(Abramov's Theorem) Suppose that $P_1 \leq P_2 \leq ... \leq N$ are countable partitions such that $H(P_n) <$ for all$n \geq 1$. Then

$$h(T) = \lim_{n \to} h(T, P_n)$$

The proof for Abramov's Theorem is quite lengthy and involved and can be found in the second source listed in the bibliography [Fen]

Next we need to introduce what a generator is. We say that a countable partition P is

Date: March 17, 2019.

a generator if T is invertible and ...

$$\bigvee_{j=-(n-1)}^{n-1} T^{-j}(P) \to N$$

as n goes to infinity. We say that a countable partition is a strong generator when ...

$$\bigvee_{j=0}^{n-1} T^{-j}(P) \to N$$

Now we can fully state and prove Sinai's Generator theorem: Suppose P is a strong generator or that T is invertible and P is a generator. If $H(P) <$ then ...

$$h(T) = H_\mu(T, P)$$

Proof. Both proofs are similar. We show the case where T is invertible and P is a generator

$$H_\mu(T, \bigvee j = -n^n T^{-j}(P)) = \lim_{k\to} \frac{1}{k} H_\mu(T^n(P) \vee ... \vee T^{-n}(P) \vee T^{-(n-1)}(P) \vee ... \vee T^{-(n+k-1)}(P)$$

$$= \lim_{k\to} \frac{1}{k} H_\mu(P \vee ... \vee T^{-(2n+k-1)}P) = H_\mu(T, P)$$

Because P is a generator ...

$$\bigvee_{j=-n}^{n} T^{-j}(P) \to N$$

and thus by Abramov's theorem,

$$h(T) = H_\mu(T, P)$$

∎

Interestingly, it can be shown that for $k \geq 0$ $h(T^k) = kh(T)$ and that $h(T) = h(T^{-1})$, both of which is also done in the second paper linked.

References

[Fen] Tony Feng. Notes on ergodic theory.
[Wal] Charles Walkden. Math41112/61112 ergodic theory.