L^p SPACES: PROPERTIES AND APPLICATIONS

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1. Introduction

L^p Spaces, occasionally referred to as Lesbegue spaces, are named after Henri Lesbegue although they were first introduced to the world from Frigyes Riesz. L^p spaces are a very important topic in mathematics, specifically analysis. This comes from the fact that L^p spaces form a very important class of both Banach and topological vector spaces. These spaces are very important in measure theory and probability theory and have many applications to theoretical physics and financial statistics.

In statistics, L^p metrics help define measures of central tendency and statistical dispersion, including mean, median, mode and standard deviation of a set of data.

L^p spaces also have great importance in Fourier Analysis. Two important theorems in Fourier Analysis are the Hausdorff-Young Theorem and the Riesz-Thorin theorem which gets us for the real line, $L^p(\mathbb{R}) \to L^q(\mathbb{R})$ when $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$ is a consequence of the Riesz-Thorin theorem of interpolation and Hausdorff-Young Inequality. We will investigate this idea a bit more throughout the paper.

Lastly, L^p spaces are very important in quantum mechanics. As I will discuss more deeply later, Hilbert and Banach Spaces are forms of all L^p spaces and L^2 spaces, respectively. A Hilbert space can generalize the notion of a Euclidean Space and extends methods of vector algebra and calculus to infinite dimensions which is very useful in quantum mechanics.

2. Preliminary Concepts — Measures and Integrals

We will begin by defining what a measure is.

Definition 2.1. A measure is a function on a set X and a σ-algebra $\mathcal{A}$, $(X, \mathcal{A})$, $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(\varnothing) = 0$ and unions of finite or countable pairwise disjoint elements of $\mathcal{A}$, then

$$\mu\left(\bigcup_n S_n\right) = \sum_n \mu(S_n).$$

Definition 2.2. A set X is measurable if you take a σ-algebra $\mathcal{S}$ where $A \subseteq X$ and $A \subseteq \mathcal{S}$.

Next, we shall define a Measure space.

Definition 2.3. Given a set X, a σ-algebra $\mathcal{A}$ and a measure μ on $(X, \mathcal{A})$ then the triple $(X, \mathcal{A}, \mu)$ is a measure space.

Definition 2.4. To continue the idea of being measurable, a space, $(X, \mathcal{A})$ can become a measure space if all of the elements of $\mathcal{A}$ are measurable sets.

Now we will introduce the concept of an outer measure, more specifically, the Lebesgue outer measure which will lead us to the Lebesgue Integrals.

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Definition 2.5. Given a set X, an outer measure on X is a function defined as $\mu^* : \mathcal{P}(x) \to [0, \infty]$. The outer measures follows the properties of monotonicity, countable subadditivity and the measure of a null set is 0.

Lastly we will introduce the Lebesgue Outer Measure before we talk about integrals.

Definition 2.6. The *Lebesgue outer measure* λ^* on $\mathbb{R}$ is defined as follows. For any $A \subseteq \mathbb{R}$, let $\mathscr{C}(A)$ denote the set of all finite or countable sequences (x_i, y_i) of open intervals in $\mathbb{R}$ such that

$$A \subseteq \bigcup_i (x_i, y_i)$$

Then define

$$\lambda^*(A) = \inf \left\{ \sum_i (y_i - x_i) : \{(x_i, y_i)\} \in \mathscr{C}(A) \right\}.$$

Now we will define a few types of functions and the Lebesgue Integral of these functions.

Definition 2.7. An indicator function on a measure space $(X, \mathcal{A}, \mu)$ of some set A such hat $A \in \mathcal{A}$, is defined to be the function

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

Definition 2.8. Let $(X, \mathcal{A}, \mu)$ be a measure space. For $A \in \mathcal{A}$, define the *(Lebesgue) integral* of $\mathbb{1}_A$ to be

$$\int_X \mathbb{1}_A \, d\mu = \mu(A)$$

Example. Let's take the indicator function of the set of rational numbers, written as $\mathbb{1}_{\mathbb{Q}}$ and $\mathbb{1}_{\mathbb{R}\setminus\mathbb{Q}}$. If we take the integral over a set X where $X = [0, 1]$ then

$$\int_X \mathbb{1}_{\mathbb{Q}} \, d\mu = \mu(\mathbb{Q}) = 0$$

and

$$\int_X \mathbb{1}_{\mathbb{R}\setminus\mathbb{Q}} \, d\mu = \mu(\mathbb{R} \setminus \mathbb{Q}) = 1$$

Next we will take a step up and learn about simple functions, basically just weighted sums of indicator functions.

Definition 2.9. Take $(X, \mathcal{A}, \mu)$ as a measure space. W define a simple function of $f : X \to \mathbb{R}$ to be a function $f(x)$ such that

$$f(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}(x)$$

where $A_1, A_2, \ldots, A_n \in \mathcal{A}$ are finitely many subsets and $c_1, c_2, ..., c_n$ are real numbers.

Now, the integral over a simple function is just an extension of the integral over an indicator function, defined to be

$$\int_X g \, d\mu = \sum_{i=1}^{n} c_i \mu(A_i)$$

where g is some simple function and we the c_i's and A_i's are what we defined using definition 2.7.

Next, let's generalize to the Lebesgue integral over any function. The *Lebesgue integral* is defined to be

$$\int_X f\,d\mu = \sup\left\{\int_X s\,d\mu : s \text{ is simple and } s(x) \le f(x) \text{ for all } x\right\}$$

This holds if $f : X \to \mathbb{R}_{\ge 0}$.

Finally, Lebesgue Integrals contain some very nice properties including monotonicity. Let's take a look at a couple other nice properties of Lebesgue Integrals.

- If f and g are simple functions, and $\alpha, \beta \in \mathbb{R}$, then we have

  $$\int_X (\alpha f + \beta g)\,d\mu = \alpha \int_X f\,d\mu + \beta \int_X g\,d\mu,$$

  Then, and if $f(x) \le g(x)$ for all $x \in X$, then

  $$\int_X f\,d\mu \le \int_X g\,d\mu$$

- The Cauchy-Schwarz inequality holds for Lebesgue integral, as

  $$\left(\int_X fg\,d\mu\right)^2 \le \left(\int_X f^2\,d\mu\right)\left(\int_X g^2\,d\mu\right)$$

As we saw earlier we have to take integrals of functions which are non-negative, but to get around that we can break up our function f into a difference of two functions where $f = f^+ - f^-$. An f^+ function is defined as $f^+(x) = \max(f(x), 0)$ while an f^- function is defined as $f^-(x) = \max(0, -f(x))$. Now we have our integral of f equal to

$$\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu.$$

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3. What are L^p Spaces

Now that we have introduced the key concepts behind L^p spaces, we can finally discuss the main topic of this paper, L^p Spaces.

Definition 3.1. An L^p Space is a function space where the p-th power of the absolute value of a function is Lebesgue Integrable with finite measure.

Please note that from now on, $(X, \mathcal{A}, \mu)$ denotes a measure space where X has finite measure, $\mathcal{A}$ denotes the σ-algebra of measurable sets and μ is a measure.

Remark 3.2. We write the L^p Spaces as either $L^p(X, \mathcal{A}, \mu)$, $L^p(X, \mu)$ $L^p(x)$ or just plainly, L^p when the measure space has been explicitly specified.

Definition 3.3. If $f \in L^p(X, \mathcal{A}, \mu)$ then f's L^p norm is defined as

$$\|f_p\| = (\int_X |f(x)|^p\,d\mu)^{\frac{1}{p}}$$

L^1 is the space of all Lebesgue-integrable functions of X.

Remark 3.4. The L^p space is a vector space, so the following properties hold:

- $(f+g)(x) = f(x) + g(x)$
- $(\lambda f)(x) = \lambda f(x)$

Definition 3.5. Essential Supremum Given a measurable function $f : X \to \mathbb{R}$, where (X, α, μ) is a measure space, the essential supremum is the smallest number α such that the set $\{x : f(x) > \alpha\}$ has measure zero. If no such number exists, as in the case of $f(x) = 1/x$ on $(0, 1)$, then the essential supremum is ∞.

Definition 3.6. The L^∞ norm is defined to be ess sup(f) where ess sup is the essential supremum.

Remark 3.7. If we integrate over $\mathbb{Z}$ with respect to the counting measure, $\mu(x) = |x|$ then we will get a discrete L^p space, where we will get measureable functions which are simply sequences $f = (a_n)_{n\in\mathbb{Z}}$ of complex numbers and we would have $\|f\|_p = (\sum_{n=-\infty}^{\infty} |a_n|^p)^{\frac{1}{p}}$.

Definition 3.8. Let p and q be two exponents between 1 and ∞. If $\frac{1}{p} + \frac{1}{q} = 1$ then p and q are referred to as dual exponents, but some people refer to them as conjugate exponents.

Using this definition we can now define Hölder's Inequality.

Theorem 3.9. ***(Hölder's Inequality)***
Let p and q be dual exponents between 1 and ∞. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and $\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q$.

In order to prove this, we first must use a lemma, Young's inequality.

Lemma 3.10. ***(Young's Inequality)***
$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$

Proof. **Proof of Hölder's Inequality**
Let's begin by calling $A = \|f\|_p$ and $B = \|g\|_q$, where neither A nor B is equal to 0 otherwise this would be extremely trivial. We want to apply Young's Inequality, so $a = \frac{|f|}{A}$ and $b = \frac{|g|}{B}$. Now we will apply Young's inequality and get $ab = \frac{|f(x)g(x)|}{AB} \leq \frac{|f(x)|^p}{pA^p} + \frac{|g(x)|^q}{qA^q} = \frac{a^p}{p} + \frac{b^q}{q}$.

$$\frac{1}{AB}\int_X |f(x)g(x)|d\mu \leq \frac{1}{pA^p}\int_X |f|^p + \frac{1}{qB^q}\int_X |g|^q d\mu$$

.

However, $A^p = \int_X |f|^p d\mu$ while $B^q = \int_X |g|^q d\mu$, therefore we get

$$\frac{1}{\|f\|_p \|g\|_q}\|fg\|_1 \leq \frac{1}{p} + \frac{1}{q} = 1$$

which leads us to

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Now, we have proved Hölder's Inequality.

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Now, we can prove the triangle inequality for L^p spaces, otherwise known as Minkowski's Inequality.

Theorem 3.11. ***Minkowski's Inequality***
If $1 \leq p \leq \infty$ and $f, g \in L^p$, then $f + g \in L^p$ and $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

Proof. We can take the $p = 1$ case to obtain $|f(x) + g(x)|\leq|f(x)|+|g(x)|$ by integrating. When $p \geq 1$, we can begin by verifying that $f + g \in L^p$ when both f and g belong to L^p. Indeed,

$$|f(x) + g(x)|^p \leq 2^p(|f(x)|^p + |g(x)|^p)$$

as can be seen by considering separately the cases $|f(x)| \leq|g(x)|$ and $|g(x)| \leq|f(x)|$. Next we note that

$$|f(x) + g(x)|^p\leq|f(x)||f(x) + g(x)|^{p-1}+|g(x)||f(x) + g(x)|^{p-1}$$

If q is the conjugate exponent of p, then $(p-1)q = p$, so we can see that $(f+g)^{p-1}$ belongs to L^q, and therefore we can apply Hölder's inequality to the two terms on the right-hand side which will give us
(Eqn. A)

$$\|f + g\|_p^p \leq \|f\|_p\|(f + g)^{p-1}\|_q + \|g\|_p|(f + g)^{p-1}\|_q$$

However, when we use $(p - 1)q = p$ again, then we will get

$$\|(f + g)^{p-1}\|_q = \|f + g\|_p^{\frac{p}{q}}$$

Taking (Eqn. A), the fact that $p - \frac{p}{q} = 1$, and we may suppose that $\|f + g\|_p \geq 0$, we find

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

And now we are finished with our proof of Minkowski's Inequality.

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4. L^p Spaces, Banach Spaces and Hilbert Spaces

In order to understand Banach spaces, we first must learn about Cauchy Sequences.

Definition 4.1. A Cauchy Sequence is a sequence whose elements become arbitrarily close to each to each other as the sequence progresses.

To specify this definition, given any small positive distance, all but a finite number of elements of the sequence are less than that given distance from each other.

Definition 4.2. A Banach space is a vector space X over a field $\mathbb{R}$ of real or $\mathbb{C}$ complex numbers, which is equipped with the norm $\|\cdot\|_X$ and which is complete with respect to the distance function induced by the norm.

To specify this, for every Cauchy Sequence x_n in X, $\exists x \in X$ so that $\lim_{n\to\infty} x_n = x$, or equivalently, $\lim_{n\to\infty}\|x_n - x\|_X = 0$.

Remark 4.3. All L^p Spaces are Banach spaces; however, not all Banach spaces are L^p Spaces.

Now, we can start to explore the idea of a Hilbert space. Similar to the relationship between L^p and Banach spaces, not all Banach spaces are Hilbert spaces, but all Hilbert spaces are Banach spaces. another space which are all Banach spaces, but not all of these spaces are Banach. In order to define a Hilbert Space we first must learn about what inner product and inner product spaces are.

Definition 4.4. An *inner product* on a complex linear space X is a map $(\cdot,\cdot) : X \times X \to \mathbb{C}$, such that $\forall x, y, z \in X$ and $\lambda, \mu \in \mathbb{C}$:

- $(x, \lambda y + \mu z) = \lambda(x, y) + \mu(x, z)$ (Linear Second Argument)
- $(y, x) = \overline{(x, y)}$ (Hermitian Symmetric)
- $(x, x) \geq 0$ (Nonnegative)
- $(x, x) = 0$ if and only if $x = 0$ (Positive definite)

Now we shall define an inner product space.

Definition 4.5. A linear space with an inner product is referred to as an *inner product space* or a *pre-Hilbert Space*.

Definition 4.6. A Hilbert Space is a complete inner product space.

To add on to what we commented earlier, every Hilbert Space is a Banach space with respect to the norm where $\|x\| = \sqrt{(x, x)}$

Remark 4.7. The L^2 Space is a Hilbert Space.

This can be shown from the L^2 Inner-Product with respect to the measure μ.

$$\langle f, g \rangle = \int_X fg d\mu$$

.

Now, the functions in an L^2 Space satisfy

$$\langle \phi \mid \psi \rangle = \int_X \overline{\psi} \phi d\mu$$

along with

- $\overline{\langle \phi \mid \psi \rangle} = \langle \psi \mid \phi \rangle$
- $\langle \phi \mid \lambda_1 \psi_1 + \lambda_2 \psi_2 \rangle = \lambda_1 \langle \phi \mid \psi_1 \rangle + \lambda_2 \langle \phi \mid \psi_2 \rangle$
- $\langle \lambda_1 \phi_1 + \lambda_2 \phi_2 \mid \psi \rangle = \overline{\lambda_1} \langle \phi_1 \mid \psi \rangle + \overline{\lambda_2} \phi_2 \mid \psi \rangle$
- $\langle \psi \mid \psi \rangle \in \mathbb{R} \geq 0$
- $\| \langle \psi_1 \mid \psi_2 \rangle \|^2 \leq \langle \psi_1 \mid \psi_1 \rangle \langle \psi_2 \mid \psi_2 \rangle$ (Cauchy-Schwarz Inequality)

5. Density and Completeness of L^p Spaces

Nos that we've discussed several interesting theorems of L^p spaces, we're going to study another aspect of L^p spaces, density and completeness.

Remark 5.1. The triangle inequality which we discussed in section 2, otherwise known as Minkowski's Inequality makes L^p into a metric space with the distance $d(f, g) = \|f - g\|_p$

There is an analytic fact that L^p spaces are complete which comes from the idea that every Cauchy sequence in the norm $\|\cdot\|$ converges to some element in L^p.
Now, let's formally prove this.

Theorem 5.2. *The space $L^p(X, \mathcal{A}, \mu)$ is complete in the norm $\|\cdot\|_p$*

Proof. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in L^p, and then take a subsequence $(f_{n_k})_{k=1}^{\infty}$ of (f_n) with the property that $\|f_{n_{k+1}} - f_{n_k}\|_p \leq 2^{-k}\ \forall k \geq 1$.

Now, let's consider some series whose convergence can be written as

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty}(f_{n_{k+1}}(x) - f_{n_k}(x))$$

and

$$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty}|f_{n_{k+1}}(x) - f_{n_k}(x)|,$$

along with the partial sums

$$S_K(f)(x) = f_{n_1}(x) + \sum_{k=1}^{K}(f_{n_{k+1}}(x) - f_{n_k}(x))$$

and

$$S_K(g)(x) = |f_{n_1}(x)| + \sum_{k=1}^{K}|f_{n_{k+1}}(x) - f_{n_k}(x)|$$

Now, we can apply Minkowski's Inequality which implies

$$\|S_K(g)\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{K}\|f_{n_{k+1}} - f_{n_k}\|_p \leq \|f_{n_1}\|_p + \sum_{k=1}^{K} 2^{-k}.$$

Now, if we take the limit as K goes to infinity and let K go to infinity and then apply the monotone convergence theorem shows us that $\int_X g^p d\mu \leq \infty$, so the series defining g, and thus the series defining f converges a.e., and $f \in L^p$.

We can now show that the desired limit of the sequence f_n is f. From this we can conclude that because the $(K-1)^{th}$ partial sum is f_{n_K},

$$f_{n_K} \to f(x) a.e.x.$$

We also know that $f_{n_K} \to f$ in L^p.

For our last step, we have to conclude that f_n is Cauchy. Given $\epsilon \geq 0, \exists N$ so that $\forall n, m \geq N$ we get $\|f_n - f_m\|_p \leq \frac{\epsilon}{2}$.

If we take some n_K s.t. $n_K \geq N$, and $\|f_{n_K} - f\|_p \leq \frac{\epsilon}{2}$, then from Minkowski's inequality we can get

$$\|f_n - f\|_p \leq \|f_n - f_{n_K}\|_p + \|f_{n_K} - f\|_p \leq \epsilon$$

whenever $n \geq N$.

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6. The Weak L^p

For our final section, we'll discuss the Weak L^p

Definition 6.1. Measurable Function
A function between two measurable spaces such that the preimage of any measurable set is measurable.

We can also add to this as analogously to our definition that a function between topological spaces is continuous if the preimage of each open set is open.

A more formal definition to this can be extended here.

Definition 6.2. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, meaning that X and Y are sets equipped with respective σ-algebras $\mathcal{A}$ and $\mathcal{B}$.

A function $f : X \to Y$ is said to be measurable if the preimage of $\mathcal{E}$ under f is in $\mathcal{A} \forall \mathcal{E} \in T$; i.e.,

$f^{-1}(\mathcal{E}) := \{x \in X \mid f(x) \in \mathcal{E}\} \in \mathcal{A}, \forall \mathcal{E} \in T$

If $f : X \to Y$ is a measurable function we will write $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ to emphasize the dependency on the σ-algebras $\mathcal{A}$ and $\mathcal{B}$.

Definition 6.3. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f be a measurable function with real or complex values on X. The distribution function f is defined $\forall t \geq 0$ by $\lambda_f(t) = \mu x \in X : |f(x)| \geq t$

Definition 6.4. Markov's Inequality If X is a nonnegative random variable and a is a random variable, then the probability that X is at least a is at most the expectation of X divided by a. $P(X \geq a) \leq \frac{E(X)}{a}$

Remark 6.5. If $f \in L^p$ for some p between 1 and ∞, then by Markov's inequality, defined earlier, $\lambda_f(t) \leq \frac{\|(f)\|_p^p}{t^p}$

Finally, we can now define what a weak L^p is.

Definition 6.6. A function f is said to be in the weak L^p, if $\exists$ a constant C such that $\forall t \geq 0, \lambda_f(t) \leq \frac{C^p}{t^p}$.

The best constant C for this inequality is the $L^{p,w}$-norm of f, which is denoted by $\|f\|_{p,w} = \sup_{t \geq 0} t \lambda_f^{\frac{1}{p}}(t)$.

Definition 6.7. A Lorentz Space on a measure space (X, μ) is the space of complex-valued measurable functions f on X such that the following quasinorm is finite.

To extend this,

$$\|f\|_{L^{p,q}(X,\mu)} = p^{\frac{1}{q}} \|t\mu|f| \geq t^{\frac{1}{p}}\|_{L^q(\mathbb{R}^+, \frac{dt}{t}}$$

where $0 \leq p \leq \infty$ and $0 \leq q \leq \infty$. Hence, when $q \leq \infty$,

$$\|f\|_{L^{p,q}}(X, \mu) = p^{\frac{1}{q}} (\int_0^\infty t^q \mu x : |f(x)| \geq t^{\frac{q}{p}} \frac{dt}{t})^{\frac{1}{q}},$$

but when $q = \infty$,

$$\|f\|^p_{L^{p,\infty}(X,\mu)} = \sup_{t \geq 0} (t^p \mu x : |f(x)| \geq t)$$

Now, we have completed a brief overview of L^p spaces and their properties. We reviewed the basic ideas of measure theory, discussed L^p spaces and theorems pertaining to L^p spaces such as Hölder, Minkowski and Young's inequalities. Furthermore, we studied the key properties of density and completeness of L^p spaces and then while learning about the weak L^p we learned about Markov's Inequality which is especially relevant to statistics.

7. References

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