THE WEAK* TOPOLOGY.

ETHAN YANG

1. Introduction

In this paper, we give background on topology in section 2. We then understand the motivation for weaker topologies in section 3 and understand Tychonoff's theorem in section 4. Finally, in section 5, we state and prove some of the main results relating to the weak* topology. In particular, we prove the Banach-Alaoglu theorem, stating that, in the weak* topology, the closed unit ball of the dual space of a normed vector space is compact. The Banach-Alaoglu theorem was first proved in the case of separable normed vector spaces by Banach in 1932, and was proved for the general case by Alaoglu in 1940.

2. Background

Definition 2.1. A *topological space* (X, T) is a set X along with a set T, called a topology on X, of subsets called the open subsets that satisfies the following:

(1) The empty set and X are in T.
(2) Any finite or countably infinite union of elements in T is still in T.
(3) Any finite intersection of elements in T is still in T.

Perhaps the simplest example we can give of a topology is to take $T = \{\varnothing, X\}$, which is clearly closed under unions and intersections.

This definition also aligns with our standard definition of open sets for the reals, where the open sets are generated by the unions and intersections of open intervals of real numbers. Similarily, we would like to generalize the concept of closed sets from the real numbers.

Definition 2.2. Let (X, T) be a topological space. A subset C of X is called closed if its complement with respect to X is open.

Furthermore, we formalize the concept of continuity into topological spaces.

Definition 2.3. Let (X_1, T_1) and (X_2, T_2) be topological spaces and $f : X_1 \to X_2$ a function from X_1 to X_2. Then f is a *continuous mapping* if for each $O \in X_2$, $f^{-1}(O) \in X_1$. That is, for each open set in the range, the preimage is open.

In a general sense, we try to expand the notion of being both closed and bounded from real numbers to general topological spaces.

Definition 2.4. A topological space X is called *compact* if each of its open covers has a finite subcover. In other words, it is compact if for every collection of open sets C such that $X = \bigcup_{x \in C} x$, then there exists a finite subset $G \subset C$ such that $X = \bigcup_{x \in G} x$.

In most cases that we are interested in, this definition coincides with another interpretation of compactness. In particular, we have the following formal definition.

Definition 2.5. A topological space X is called *sequentially compact* if every sequence of points in X has a subsequence that is convergent to some point in X.

The Bolzano-Weirstrass theorem further connects these concepts by stating any closed and bounded subset of the real numbers is also sequentially compact. In general topological spaces, these two concepts are not exactly related. However, it turns out that, in a metric space, sequentially compact and compact are equivalent.

Finally, we give the definition of convergence in a topological space.

Definition 2.6. Let (X, T) be a topological space and (x_i) be a sequence of points in X. Then (x_i) converges to x if for every open neighborhood U of x there exists a $N \in \mathbb{N}$ such that for every $n \geq N$, $x_n \in U$.

As expected, this definition coincides when we use the real numbers as our set with the standard topology. The open neighborhoods become open intervals around x that are of ε length.

3. Weaker and Stronger Topologies

Weaker and stronger are notions of containment in topology. This allows us to determine properties in our original topologies by proving something weaker.

Definition 3.1. Let X be a topological space and $T_1 \subset T_2$ be topologies on X. Then we call T_1 to be *weaker* than T_2. Similarily, T_2 is called *stronger* than T_1.

Our motivation behind finding a weaker topology is that certain properties may be easier to establish in one topology, but we can then bring those properties into a stronger or weaker topology. Specifically, openness and closedness in the weaker topology implies it in the stronger one. However, in the other direction, convergence and compactness in the stronger topology implies it in the weaker. We state these propositions formally and prove them.

Proposition 3.2. *Let X, T_1, and T_2 be as the above definition. If O is open with respect to T_1, then O is open with respect to T_2.*

Proof. Since $O \in T_1$, O is also in T_2 and thus open with respect to T_2. ■

Proposition 3.3. *Let X, T_1, and T_2 be as the above definition. If C is closed with respect to T_1, then C is closed with respect to T_2.*

Proof. The complement of C must be in both T_1 and T_2. Thus, C is closed in T_2. ■

Proposition 3.4. *Let X, T_1, and T_2 be as the above definition. If S is compact with respect to T_2, then S is compact with respect to T_1.*

Proof. Assuming S is compact with respect to T_2, we need to prove that S is also compact with respect to T_1 by proving any open cover from T_1 has a finite subcover. Take some open cover C with open sets from T_1. Then, it is also an open cover with respect to T_2. Since S is compact with respect to T_2, there is a finite subcover $G \subset C$. Thus, S is also compact with respect to T_1. ■

Proposition 3.5. *Let X, T_1, and T_2 be as the above definition. If the sequence (x_i) converges to x in T_2, then (x_i) also converges to x in T_1.*

Proof. Take some open neighborhood $U \in T_1$ of x. Since T_2 is stronger than T_1, we have that $U \in T_2$. Since (x_i) is convergent in T_2, we have that, for some $N \in \mathbb{N}$, every $n \geq N$, $x_n \in U$. Thus, (x_i) is convergent to x in T_1 as well. ■

4. Tychonoff's Theorem

Before we prove the Banach-Anaoglu theorem, we need to state Tychonoff's theorem. Tychonoff's theorem makes a statement about the product topology.

Definition 4.1. Let (X_i, T_i) be topological spaces indexed by $i \in I$. Let $X = \prod_{i \in I} X_i$, or the cartesian product of the topological spaces X_i. The *product topology* T on the set X is the weakest topology such that all the canonical projections $p_i : X \to X_i$ are continuous. It can be shown that the product topology is generated by the set

$$B = \left\{ \prod_{i \in I} O_i : O_i \in T_i \text{ and } O_i = X_i \text{ for all but a finite amount of } i \right\}.$$

We use an arbitrary set I to index our spaces in order to be allowed to define the topology for the product of uncountably many spaces. This definition aligns with how we define the Euclidean topology on $\mathbb{R}^n$. For $\mathbb{R}^2$, open intervals are replaced with open rectangles - rectangles with open borders. This makes sense, as an open rectangle is exactly the cartesian product of two open intervals. Thus any open set for $\mathbb{R}^2$ is generated by a union of some open rectangles.

Theorem 4.2 (Tychonoff's Theorem)**.** *The product of any collection of compact topological spaces is compact in the product topology.*

This theorem is extremely powerful, as it provides an incredibly general way to produce compact sets in the product topology. It is applied in the proofs of many theorems that want to prove compactness in a relatively general sense.

There exists many different proofs of Tychonoff's theorem, such as ones that use ultrafilters and others that use nets. However, all proofs of Tychonoff's theorem use the axiom of choice in some way. For example, the proof involving ultrafilters also uses Zorn's Lemma, which is proved from the axiom of choice. It turns out that the converse is also true, where Tychonoff's theorem implies the axiom of choice.

5. The Weak* Topology

We then move to start to define both the weak and the weak* topologies. Before that, we also want the topological spaces we are interested to have our standard operations to be well-defined.

Definition 5.1. A *topological field* is a field with a topological space such that addition, multiplication, and division are continuous.

For example, the complex numbers and the real numbers are both topological fields with respect to their normal topologies.

Definition 5.2. A *topological vector space* is a vector space over a topological field with a topology such that both vector addition and scalar multiplication are continuous.

It turns out that there is a natural and typical example of a topological vector space.

Definition 5.3. A *normed vector space* is a vector space on which a norm that assigns reals to vectors is defined. The norm must satisfy:

(1) $||x|| \geq 0$, except for the zero vector where $||0|| = 0$.
(2) $||\alpha x|| = |\alpha| ||x||$.
(3) The triangle inequality holds: $||x + y|| \leq ||x|| + ||y||$.

Our typical example of a norm is the standard Euclidean length of vectors in $\mathbb{R}^n$.

Furthermore, a norm provides us a metric $\mu(x, y) = ||x - y||$. This metric based on the norm induces a simple topology we can define on the normed vector space, making it a topological vector space.

Definition 5.4. The *norm topology* or *strong topology* on a normed vector space X has the open sets that are generated by the open balls $B_x = \{y \in X : ||y - x|| < r\}$ for all $x \in X, r \in \mathbb{R}$. In other words, every open set can be written as the intersection and union of some open balls.

However, as the name implies, this topology is often too strong and thus very few sets are compact in this topology.

Definition 5.5. A *linear functional* of a vector space V over a field K is the set of all functions $f : V \to K$ that satisfy the following:

(1) $f(v + w) = f(v) + f(w)$ for all $v, w \in V$.
(2) $f(av) = af(v)$ for all $a \in K, v \in V$.

In the case of vectors in $x = \langle x_1, x_2, \ldots, x_n \rangle \in \mathbb{R}^n$, every linear functional can be represented as $f(x) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$, where $a_i \in \mathbb{R}$.

Definition 5.6. The *continuous dual*, or dual space, of a topological vector space V, typically denoted by V^* is the vector space of continuous linear functionals of V using addition and scalar multiplication as its operations.

Definition 5.7. Let X be a normed vector space and its dual X^*. The *weak topology* on X is the intersection of all the topologies such that each element of X^* remains continuous. The weak* topology on X^* is the intersection of all the topologies such that every element in X corresponds to a continuous map to X^*.

Often, we want to determine whether a set is compact or not in some topological space. However, the standard definition of compactness is hard to check in most topological spaces. Ideally, we would like something similar to the real numbers, where a compact set is the same as sequentially compact and also the same as closed and bounded. The Banach-Alaoglu theorem provides us a guide for a similar idea to find compactness in a general toplogical space.

The basic sketch of the proofs is just to use the direct product topology and to apply Tychonoff's theorem on the product of closed disks. The unit ball is a closed subset of this direct product and the topology induced by it is exactly the weak* topology. As a result of the use of Tychonoff's theorem, the proof of this requires the axiom of choice and relies on the ZFC axioms. However, we must prove the following standard lemma about subsets of compact sets.

Lemma 5.8. *Let T be a compact topological space. If $S \subset T$ such that S is closed, then S is compact.*

Proof. Let U be an open cover of S. Since S is closed, we know that S^C is open. Thus, $S^C \cup U$ is an open cover of T. By the compactness of T, there is a finite subcover of $S^C \cup U$ and thus a finite super cover of U. ■

Theorem 5.9 (Banach–Alaoglu Theorem)**.** *Let X be a normed vector space and its dual X^*. The closed unit ball B of X^* is compact with respect to the weak* topology.*

Proof. Let

$$Y_x = \{z \in \mathbb{C} : ||z|| \leq 1\}, Y = \prod_{x \in X} Y_x.$$

Clearly, each Y_x is compact. By Tychonoff's theorem, Y must also be compact in the product topology. Note that $B \subset Y$ because every element of B is a mapping to $b \in \mathbb{C}$ from $x \in X$ with $|b(x)| \leq ||x||$. Furthermore, the topology induced by Y on B is exactly the weak* topology, and in particular there is not a weaker topology. Finally, we must show that B is closed under Y. We provide a brief overview for how to do this, and the rest is left to the reader. Ideally, we would like to use limits of sequences to prove closedness. It can be quickly shown that the limits of sequences preserves the linearity of linear functionals. However, our other assumption that these sequences could prove closedness is too good to be true. Instead, the concepts of generalized sequences, or nets, must be developed. ■