HAAR MEASURE

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1. Introduction

Introduces the motivation for the Haar measure: measure with certain "nice" properties across all groups. Specifying what types of groups we want (must be a topological group for measure to make sense, and must be Hausdorff and locally compact).

Measure theory gives us a good generalization of lengths to all sets; however, it requires that we actually find a measure over a given set first. Over the real numbers, we have the Lebesgue measure; however, the measure is not obviously easy to transfer to other types of sets.

We seek a way to generalize the Lebesgue measure to all groups (for which the concepts we are describing make sense). First, we specify what exactly this means.

Definition 1.1. A *Topological Group* is a group with a topology defined over it such that the group operation is continuous and the "inverse function" $f(x) = x^{-1}$ is also continuous.

This is the most basic condition we need satisfied for measures to make sense at all. But there are some other nice properties we'd like before going forward.

Definition 1.2. A space X is *Hausdorff* if for all $x, y \in X, x \neq y$, there exist open sets U, V such that $x \in U, y \in V, x \cap y = 0$.

In other words, a space is Hausdorff if all pairs of distinct points are separated by open sets.

Definition 1.3. A measure space M is a topologial measure space is a measure space (X, Σ, μ) such that X is a topological space.

Thus, we can now more precisely define what we want: A measure over a topological group that is locally compact and Hausdorff. Or, more specifically, a measure that exists over *every* locally compact Hausdorff topological group.

2. The Haar measure

The measure that satisfies these criteria is the Haar measure. To define a Haar measure, we must first give the defnition of a regular measure.

Definition 2.1. A measure over a Hausdorff topological space is *regular* iff: 1. All compact sets have finite measure 2. The measure of any set is the infimum of the measures of all open sets containing the set. 3. The measure of any open set is the supremum of the measures of its compact subsets.

With this definition, we can now define what it means for a measure to be Haar over a topological group.

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Definition 2.2. A left Haar measure over a locally compact Hausdorff topological group G is a regular, nonzero measure μ such that $\mu(gA) = \mu(A)$ for all subsets A of G and all $g \in G$.

In other words, the Haar measure is invariant over the group operation for all elements of the group.

Remark 2.3. The *right Haar measure* is defined the same as the left Haar measure except that $\mu(Ag) = \mu(A)$ instead of $\mu(gA)$.

This generalizes a nice property over the real numbers: translation invariance. For a good measure over the reals, we want the measure of $B = \{a + k \mid a \in A\}$ to be the same as the measure of A. Thus, the Haar measure over G is a generalization of the Lebesgue measure over $\mathbb{R}^+$, which suggests the following:

Proposition 2.4. *For the group $\mathbb{R}$ under addition, the Lebesgue measure is a Haar measure.*

Proof. Firstly, $\mathbb{R}^+$ satisfies the requisite conditions for having a Haar measure defined over it: it is a topological space that is both Hausdorff and locally compact. In addition, the Lebesgue measure is regular by its definition.

We know that the Lebesgue measure of any measurable set can be constructed from the lengths of intervals which cover it. Since interval length is translation invariant, the Lebesgue measure is also translation invariant. ■

3. Equivalence of left and right Haar measure

Remark 3.1. In the following proofs, the σ-algebra generated by a subset S of the power set shall be denoted $\sigma[S]$.

First, we show that a right Haar measure can be constructed from a left Haar measure and vice versa. To do this, we need the following lemma:

Lemma 3.2. *Given a homeomorphism $f : X \to X$ of topological measure space (X, Σ, μ), if $A \in \Sigma$, then $f(A) \in \Sigma$ and $f^{-1}(A) \in \Sigma$.*

Proof. Given a subset E of the power set, we shall show that $\sigma[f^{-1}(E)] = f^{-1}(\sigma[E])$; the lemma statement follows trivially from this.

For any $A \in f^{-1}(E)$, there exists a $B \in E$ with $A = f^{-1}(B)$. Since $B \in E, B \in \sigma[E]$, and $f^{-1}(E)$ is contained in $f^{-1}(\sigma[E])$. Since the inverse function f^{-1} is interchangable with the union and complement operations, $f^{-1}(\sigma[E])$ is closed under complementation and countable union, making it a σ-algebra that contains $f^{-1}(E)$, so $\sigma[f^{-1}(E)] \subseteq f^{-1}(\sigma[E])$.

For the other direction, we define Σ to include all A with preimages in $\sigma[f^{-1}(E)]$. For any $A \in E$, $f^{-1}(A) \in f^{-1}(E) \subseteq \sigma[f^{-1}(E)]$, so $A \in \Sigma$, meaning that $E \subseteq \Sigma$. As in the above section, f^{-1} is interchangable with unions and complements, meaning that Σ is a σ-algebra containing E, so $\sigma[E] \subseteq \Sigma$.

Taking some A in $f^{-1}(\sigma[E])$, $A = f^{-1}(B)$ for some $B \in \sigma E$. Thus $B \in \Sigma$, and, by the definition of Σ, $A \in \sigma[f^{-1}(E)]$. Thus $f^{-1}(\sigma[E]) \subseteq \sigma[f^{-1}(E)]$, and $\sigma[f^{-1}(E)] = f^{-1}(\sigma[E])$.

From this, because a topological measure space (X, Σ, μ) has $\Sigma = \sigma[\tau]$ where τ is the topology, we know that $f^{-1}(\Sigma) = \Sigma$, and the lemma follows. ■

With this, we can now define a measure μ' on G given left Haar measure μ, which (following from the lemma above) exists on the same σ-algebra as μ, and is a right Haar measure on G.

Lemma 3.3. *If μ is a left Haar measure over a topological group G, then $\mu'(A) = \mu(A^{-1})$ is a right Haar measure over G.*

Proof. The measure μ' exists over the same σ-algebra as μ per the previous lemma (as $f(g) = g^{-1}$ is a homeomorphism on G). We must next prove that μ' is regular:

1. $K \subseteq G$ compact $\iff$ K^{-1} compact, so $mu'(K) < \infty$.

2. The inverse of an open set is also open, so if A^{-1} is contained by open set U, then A is contained by open set U^{-1}. Thus, we have $\inf_{A \subseteq U^{-1}} \mu'(U^{-1}) = \inf_{A^{-1} \subseteq U} \mu(U) = \mu(A^{-1}) = \mu'(A)$.

3. Likewise, for an open set A, over all compact subsets K^{-1}, we have $\sup_{K^{-1} \subseteq A} \mu'(K^{-1}) = \sup_{K \subseteq A^{-1}} \mu(K) = \mu(A^{-1}) = \mu'(A)$.

All that remains is the right Haar condition itself.

We have:

$$\mu'(Ag) = \mu((Ag)^{-1}) = \mu(g^{-1}A^{-1}) = \mu(A^{-1}) = \mu'(A).$$

■

Thus, left and right Haar measure are equivalent; the existence and uniquness of a left Haar measure implies the same for right Haar measure. For the remainder of this paper, ”Haar measure” shall refer exclusively to the left Haar measure unless otherwise indicated.

4. Existence and Uniqueness

First, a lemma:

Lemma 4.1. *Given an open subset U of topological group G which in turn has compact subset K. Then, there exists some open set V, with the identity $e \in V$, such that $KV \subseteq V$.*

Proof. For each element $x \in K$, take $W_x = x^{-1}U$. Then take V_x such that $V_xV_x \in W_x$. The collection xV_x for all x covers K, so there exists some finite cover within that collection. Define V as the intersection of all V_x in that finite cover.

Then, for any $x \in K$, one of the x_k used in the finite cover satisfies $x \in x_kV_{x_k}$. This gives us

$$xV \subseteq x_kV_{x_k}V \subseteq x_kV_{x_k}V_{x_k} \subseteq x_kW_{x_k} = U,$$

yielding $KV \subseteq U$. ■

Theorem 4.2. *For all locally compact Hausdorff topological groups, there exists a Haar measure over that group.*

Proof. We first define (for subsets K and V of G with K compact and V having nonempty interior) $(K : V)$ as the minimum number of elements g_k of G necessary to form an open cover of K from the union of sets g_kV^o (where V^o is the interior of V).

We can then define the function μ_U, for an open subset U of G containing the identity, which maps compact subsets of G to $\mathbb{R}$ as follows:

$$\mu_U(G) = \frac{(K : U)}{(K_0 : U)}$$

where K_0 is a constant compact subset of G with nonempty interior.

From here, we can take the cartesian product of intervals $X = \Pi_{K \in \mathcal{K}}[0, (K : K_0)]$ (where $\mathcal{K}$ is the set of all compact subsets of G. Then, we know that $\mu_U(K) \leq (K : K_0)$, because a covering of K by U can be generated using the products of each of the elements used to cover K by K_0 and K_0 by U. This means that for each open U containing the identity, μ_U can be mapped to a point in X.

Then, given some open set V containing the identity, we define $C(V)$ as the closure in X of all μ_U for $U \subseteq V$. We then look at the collection of all $C(V)$. Since the intersection of a finite number of open sets containing the identity must itself contain an open set containing the identity, this collection satisfies the finite intersection property. Any collection in a compact space that satisfies the finite intersection property must have nonempty intersection, so there is some $\mu \in X$ which is contained in every $C(V)$. We select this μ as our measure, and shall prove that it is Haar.

First, we know that given $K_1 \subseteq K_2$, any covering of K_2 is also a covering of K_1. Thus, $(K_1 : U) \leq (K_2 : U)$ and μ_U. Due to how we defined X, we can consider it to contain functions with $\mathcal{K}$ as their domain. Thus, we can consider the map from X to G which takes f to $f(K)$; since $f(K)$ is just one of the coordinates of f, this map is continuous. Thus, the map of f to $f(K_2)-f(K_1)$ is also continuous. This means that $f(K_2)-f(K_1)$ is nonnegative not only on all μ_U, but on all $C(V)$ as well. Thus, $\mu(K_2) - \mu(K_1) \geq 0$, and

$$\mu(K_1) \leq \mu(K_2).$$

. We can do something similar for the union rule. $K_1 \cup K_2$ can be covered by combining the covers of K_1 and K_2, so we know that $\mu_U(K_1 \cup K_2) \leq \mu_U(K_1) + \mu_U(K_2)$. The map of f to $f(K_2) + f(K_1) - f(K_1 \cup K_2)$ is continuous. Thus, it is nonnegative on all $C(V)$, yielding $\mu(K_2) + \mu(K_1) - \mu(K_1 \cup K_2) \geq 0$ and

$$\mu(K_1 \cup K_2) \leq \mu(K_1) + \mu(K_2).$$

Next, we prove equality with a null intersection. First, given a μ_U, we take K_1, K_2 such that $K_1U^{-1} \cap K_2U^{-1} = \emptyset$. Then we take the minimum set of $g_1, g_2, g_3, \ldots, g_n$ such that the g_kU cover $K_1 \cup K_2$, meaning that $n = (K_1 \cup K_2 : U)$. No g_kU can intersect both K_1 and K_2, as that would yield $g_lk \in K_1U^{-1} \cap K_2U^{-1}$. This means that the cover can be split into two separate covers of K_1 and K_2. But $(K_1 \cup K_2 : U)$ can be no greater than $(K_1 : U) + (K_2 : U)$, the sum of the sizes of the minimal coverings of K_1 and K_2. This means that the split covers of K_1 and K_2 must be minimal coverings, and $(K_1 \cup K_2 : U) = (K_1 : U) + (K_2 : U)$. This in turn yields $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$. Now, we must prove the same for μ given $K_1 \cap K_2 = \emptyset$. Then we find disjoint open $U_1 \supseteq K_1$ and $U_2 \supseteq K_2$. By Lemma 4.1, there are open V_1 and V_2 containing the identity such that $K_1V_1 \subset U_1$ and $K_2V_2 \subset U_2$. Taking $V = V_1 \cap V_2$, we have that K_1V and K_2V are disjoint. Thus K_1U and K_2U are disjoint for all open $U \subseteq V$ containing the identity, and for all such U, $\mu_U(K_1 \cup K_2) = \mu_U(K_1) + \mu_U(K_2)$. Since we already know that $f(K_2) + f(K_1) - f(K_1 \cup K_2)$ is continuous, if it is equal to 0 for all μ_U, $U \subseteq V$, it must also be 0 for all of $C(V)$, including μ. Thus,

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$$

if $K_1 \cap K_2 = \emptyset$.

With these established, we can now extend μ beyond just compact sets K.

For an open set U, $\mu(U) = \sup\{\mu(K) | K \in \mathcal{K}, K \subseteq U\}$. We can verify that this definition is consistent when U is compact; $K \subseteq U$ implies that $\mu(K) \leq \mu(U)$ (Trivially, this definition retains thi property and all other measure properties proved above). Since U is one of the valid K, the value of the supremum must be $\mu(U)$ under the initial definition is well.

Then we can generalize further. For a subset $A \subseteq G$, $\mu(A) = \inf\{\mu U | A \subseteq U, U\text{open}\}$. We can likewise verify this definition for A open: $\mu A \leq \mu U$, but A is a valid U, so the infimum must be equal to $\mu(A)$ under the open-set definition.

This final version of μ is regular by definition. In addition, since we have defined μ_U based on a constant set K_0, we know that $\mu_U(K_0) = 1$ for all U. Since $f \to f(K)$ is continuous, $f(K_0) = 0$ over all $C(V)$ and $\mu(K_0) = 1$, meaning that μ is a nonzero measure.

Finally, all that remains to be proven is that μ is translation invariant.

If $x_1, x_2, \dots x_n$ cover K, then $gx_1, gx_2, \dots, gx_n$ cover gK, yielding

$$(K : U) = (gK : U) \implies \mu_U(K) = \mu_U(gK)$$

for all μ_U. We have already established that $f \to f(K_2) - f(K_1)$ is continuous, which means that $f(K) = f(gK)$ in all $C(V)$, and

$$\mu(K) = \mu(gK).$$

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The proof of the uniqueness of Haar measure (up to scaling) is somewhat more complicated and requires several theorems which will not be proved here.

Theorem 4.3. *Given a locally compact Hausdorff topological group G and Haar measures μ and μ' over G, $\mu' = k\mu$ for some positive real constant k.*

Remark 4.4. Throughout this proof, references to "measure" are respective μ unless otherwise specified.

Proof. Since Haar measures are nonzero, there must be some subset of G with positive measure. By regularity, this also means that there must be some compact set with positive measure, which we call K.

We then take a nonnegative function f other than the zero function which is continuous with compact support. Then, we define U as the preimage of the positive reals under f. Since f is not the zero function, U is nonempty and open, so K can be covered by a finite number n of cosets of U. Thus, we have that $\mu(K) \leq n\mu(U)$, so $\mu(U) > 0$. This, in turn, means that there is some $a > 0$ for which the set A of all $g \in G$ with positive $f(g)$ has positive measure. The Lebesgue integral of f over A must be at least $a\mu(A)$; since f is nonnegative, this also means that $\int_G f d\mu > 0$.

We now define the function $h(x, y)$ as follows:

$$h(x, y) = \frac{f(x)g(yx)}{\int_G g(tx)d\mu'(t)}.$$

It must then be proven that h is continuous; this will not be done here. Once continuity is proven, however, we can apply Fubini's theorem to the double integral

$$\int_G \int_G h(x, y)d\mu'(y)d\mu(x)$$

. This, along with several further manipulations, eventually yields the equation

$$\int_G f(x)d\mu(x) = \left(\int_G g(x)d\mu(x)\right)\left(\int_G \frac{f(y^{-1})}{\int_G g(ty^{-1})d\mu'(t)}d\mu'(y)\right).$$

Since the second term on the right hand side does not depend on μ at all, this can be rewritten as

$$\int_G f(x)d\mu(x) = C\int_G g(x)d\mu(x)$$

where C is constant.

Likewise,

$$\int_G f(x)d\mu'(x) = C\int_G g(x)d\mu'(x).$$

Since g is fixed, this yields

$$\int_G f d\mu' = a\int_G f d\mu$$

$$\int_G f d\mu = \int_G f d\left(\frac{1}{a}\mu'\right).$$

With this last equation, we can use the Riesz-Markov-Kakutani representation theorem, which states that over a locally compact Hausdorff space H, a functional over $C_C(x)$ has a unique regular measure for which it is equal to the Lebesgue integral over H for every function in $C_C(x)$. This means that the measures μ and $\frac{1}{a}\mu'$ are the same, or

$$\mu' = a\mu.$$

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