Baire Category Theorem

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1 Introduction

(X, τ) be a topological space and A be a subset of X, then the closure of A is denoted by $\overline{A}$ is the intersection of all closed sets containing A or all closed super sets of A; i.e. the smallest closed set containing A.

Let X be a metric space. The closure of set A is the smallest closed set containing A. A subset $A \subseteq X$ is called nowhere dense in X if the interior of the closure of A is empty. A is nowhere dense if and only if it is contained in a closed set with empty interior. Therefore, the complement of A contains a dense open set.

1.1 Proposition 1.1

Let X be a metric space. Then:

- Any subset of a nowhere dense set is nowhere dense.
- The closure of a nowhere dense set is nowhere dense.
- The union of finitely many nowhere dense sets is nowhere dense.
- If X has no isolated points, then every finite set is nowhere dense.

Proof. To prove the third bullet point, consider a pair of nowhere dense sets A_1 and A_2, and prove that their union is nowhere dense. It is also convenient to pass to complements, and prove that the intersection of two dense open sets V_1 and V_2 is dense and open (why is this equivalent?). It is trivial that $V_1 \cap V_2$ is open, so let us prove that it is dense. A subset is dense every nonempty open set intersects it. So fix any nonempty open set $U \subseteq X$. Then $U_1 = U \cap V_1$ is open and nonempty (why?). And by the same reasoning, $U_2 = U_1 \cap V_2 = U \cap (V_1 \cap V_2)$ is open and nonempty as well. Since U was an arbitrary nonempty open set, we have proven that $V_1 \cap V_2$ is dense. To prove the fourth bullet point, it suffices to note that a one-point set $\{x\}$ is open if and only if x is an isolated point of X; then use the second bullet point.

1.2 Baire Category Theorem

X is a complete metric space and $A1, A2, A3, \dots$ is a sequence of dense open sets. Then $\bigcap_{n=1}^{\infty} A_n$ is nonempty.

1.3 Proof of Baire Category Theorem

Pick an open ball whose closure $\overline{B_{\epsilon_1} x_1}$ is contained in A_1. Since A_2 is open and dense, $\overline{B_{\epsilon_1} x_1} \cap A_2$ contains some open ball with closure $\overline{B_{\epsilon_2} x_2}$. Since A_3 is open and dense, $\overline{B_{\epsilon_2} x_2} \cap A_3$ contains some open ball with closure $\overline{B_{\epsilon_3} x_3}$. Carry on in this process, obtaining open balls $\overline{B_{\epsilon_n} x_n}$. such that $\overline{B_{\epsilon_n} x_n} \cap A_{n+1}$ contains $\overline{B_{\epsilon_1} x_1}$ for $n = 1, 2, \dots$ Then $(x_n)^{\infty}{}_{n=1}$ is Cauchy. Since X is complete, there is $x \in X$ such that $x_n \to x$. Now for each n the centres $\underline{x_j\ ,\ j \geq n}$, they all lie in $B_{\epsilon_n} x_n$ by construction, and so x lies in the closure $\overline{B_{\epsilon_n} x_n}$ and thus in A_n. Then $\bigcap_{n=1}^{\infty} A_n$ is nonempty.

1.4 Another Proof Baire Category Theorem

We assume none of the sets A_n contain a nonempty open subset and construct a Cauchy sequence that converges to a point, which lies in none of the A_n. By assumption A_1 does not contain a nonempty open set, and therefore its open complement A_1^c must. In other words there must exist $x_1 \in X$ and $0 < \epsilon_1 < 1$ such that $B_{x_1}\epsilon_1 \subset A_1^c$ By assumption, A_2 does not contain a non-empty set, and therefore there must be a point x_2 in the open set $B_{x_1}\epsilon_1 \cap A_2^c$. Thus $\epsilon_1 < \frac{1}{2}$ is true when $B_{x_2}\epsilon_2 \subset (A_2^c \cap B_{x_1}\epsilon_1)$. Continuing this we can make a sequence of x_n where n is greater or equal to 1 and positive reals number ϵ_n where n is greater than or equal to one such that $B_{x_{n+1}}\epsilon_{n+1} \subset (A^c{}_n \cap B_{x_n}\epsilon_n)$ and $\epsilon_n < \frac{1}{2^n}$. Notice by our construction $B_{x_\infty}\epsilon_\infty \subset \cdots \subset B_{x_1}\epsilon_1$ Here we have a sequence of nested balls. The sequence x_n where n is greater than or equal to one is Cauchy since $n, m \geq N$ implies that x_n, $x_m \in B_{x_N}(\epsilon_N)$ and $(\epsilon_N) < \frac{1}{2^N}$. The definition of a Cauchy sequence is that $|x_m - x_n| < \epsilon$ for $m, n > N$. Hence by completeness there exists a point $x_\infty \in X$ such that $x_n \to x_\infty$. In particular, $x_\infty \in B_{x_N}(\epsilon_N)$ and therefore x_∞ does not belong to A_n for all $n \in \mathbb{N}$. Thus $\bigcap_{n=1}^{\infty} A_n$ is nonempty.