Spectral Techniques

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1 General Definitions

Definition 1.1. A *measure-preserving system* (X, β, μ, T) is a measure space together with a measure-preserving transformation. Here, (X, β, μ) is the measure space, where X is a set, β is the set of measurable subsets of X, and μ is the measure on X, a function $\mu : \beta \to \mathbb{R}$. T is then a function $T : \beta \to \beta$ that is measure-preserving. [1]

Definition 1.2. A transformation matrix $U \in \mathbb{C}$ has an *eigenvalue* λ, and corresponding eigenfunction $\mathbf{u}$ if $\lambda\mathbf{u} = U\mathbf{u}$. [2]

Definition 1.3. A *compact space* is a space that includes the limits of all sequences of elements in the space, and is bounded (all points lie within a certain finite distance of each other). [3]

Definition 1.4. A *metric space* is a set of elements along with a metric that defines the distance between any elements in the set. [4]

Definition 1.5. An *isometric* measure-preserving system is a measure-preserving system $(Y, \mathcal{C}, \nu, S)$ where Y is a compact metric space, $\mathcal{C}$ is the Borel σ-algebra, and S preserves the metric. [1]

Definition 1.6. A *non-trivial* measure-preserving system is one where ν is not a single atom. [1]

Definition 1.7. A *Cauchy sequence* is a sequence whose elements become arbitrarily close to each other, so for any given distance, only a finite number of elements are greater than or equal to that distance apart. [5]

Definition 1.8. A *complete* metric space is a metric space M where every Cauchy sequence of points in M converges in M. [6]

2 Ergodicity

Definition 2.1. A *Hilbert space*, H, is a complete vector space with an inner product between every two vectors. It is a generalization of Euclidean space. [7]

Definition 2.2. An *induced unitary operator* T on the Hilbert space has the property that for any $< x, y >_H$, $< T(x), T(y) >_H = < x, y >_H$. [1]

Definition 2.3. An eigenvalue λ is a *simple eigenvalue* if the corresponding eigenspace has a complex dimension of 1. [1]

Lemma 2.4. A unitary operator U only has eigenvalues of modulus 1.

We can see that if $U\mathbf{u} = \lambda\mathbf{u}$ and $\mathbf{u}$ has modulus 1, then

$$\lambda\overline{\lambda} =< \lambda\mathbf{u}, \lambda\mathbf{u} >=< U\mathbf{u}, U\mathbf{u} >=< \mathbf{u}, \mathbf{u} >= 1.$$

[1]

Theorem 2.5. Given a measure-preserving system $(X, \mathcal{B}, \mu, T)$ where T is the induced unitary operator on $L^2(\mu)$, T is ergodic if and only if 1 is a simple eigenvalue. [1]

We can begin by seeing that if T is ergodic, then for any T-invariant set A, we have that $T^{-1}(A) = A$ almost everywhere. Since T is a unitary operator, it is bijective, and $T(A) = A$. Treating A as an eigenvector means it is an eigenvector with eigenvalue 1 and since the set of eigenvectors A include ones that represent the constant function, we know that the eigenspace is at least one-dimensional. In the other direction, we can see that if the eigenspace has complex dimension of more than 1, then there exists an eigenfunction f with eigenvalue 1 and complex dimension of more than 1, which means $0 < \mu(A) < \mu(X)$. Then, we can see that since it has eigenvalue 1, $T(A) = A$, and since T is a unitary operator, $T^{-1}(A) = A$, and T must therefore not be ergodic.

Lemma 2.6. A measure-preserving system $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if every measurable invariant function is constant almost everywhere.

We know that a measure-preserving system $(X, \mathcal{B}, \mu, T)$ is ergodic if and only if every T-invariant set A has measure 0 or 1, and this produces a T-invariant function, $\nVdash_A$, that is either constant almost everywhere at 1 or 0. In addition, if we have a T-invariant function that is not constant almost everywhere, then we can find a measurable set B in the range of f such that $\mu(f^{-1}(B)) \neq 0, 1$, and the measure-preserving system is therefore not ergodic.

Corollary 2.6.1. If T is ergodic then all eigenfunctions are constant.

Since all eigenvalues of T have modulus 1, we can describe an eigenvalue $\lambda = e^{2i\pi\alpha}$. Then, we can define f to be the corresponding eigenfunction, and we have that

$$T|f| = |T(f)| = |e^{2i\pi\alpha} f| = |f|.$$

This means that $|f|$ is an invariant function, and since T is ergodic this means that $|f|$ must be constant. [1]

3 Weak Mixing

Lemma 3.1. A measure-preserving system $(X, \mathcal{B}, \mu, T)$ is weak mixing if $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu, T \times T)$ is ergodic. [1]

We can see that since $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu, T \times T)$ is ergodic, $(\mu \times \mu)((T \times T)^i(A \times A) \cap B \times B)$ weakly Césaro converges to $\mu \times \mu(A \times A)\mu \times \mu(B \times B)$ for all A and B in $\mathcal{B}$. This means

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (\mu \times \mu)((T \times T)^i(A \times A) \cap B \times B) = \mu \times \mu(A \times A)\mu \times \mu(B \times B).$$

This can then be written as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu^2(T^i(A) \cap B) = \mu^2(A)\mu^2(B).$$

Then, we can see that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (\mu(T^i(A) \cap B) - \mu(A)\mu(B))^2 = 0$$

so $(X, \mathcal{B}, \mu, T)$ must be weakly mixing.

Definition 3.2. A measure-preserving system $(Y, \mathcal{C}, \nu, S)$ is a *factor* of $(X, \mathcal{B}, \mu, T)$ if there exist $X_0 \in \mathcal{B}$ and $Y_0 \in \mathcal{C}$ such that $\mu(X_0) = \mu(X)$, $\nu(Y_0) = \nu(Y)$, and there exists a measurable map $f : X_0 \to Y_0$ such that $f(\mu) = \nu$ and $f(T) = S(f)$. Then, f is known as a *factor map.* [1]

Theorem 3.3. Let $(X, \mathcal{B}, \mu)$ be a Borel measure space and T be an invertible measure-preserving function. Then it is weak mixing if and only if it does not have non-trivial isometric factors.

We can see that if X is weak mixing, then it does not have non-trivial isometric factors, which we can prove by contradiction. If X has a non-trivial isometric factor Y then $X \times X$ has a non-trivial isometric factor $Y \times Y$. Then, we can see that since d, the metric of $Y \times Y$ is a non-trivial invariant function $d : Y \times Y \to \mathbb{R}$, $Y \times Y$ is not ergodic. This means $X \times X$ is not ergodic either, and X then cannot be weak mixing, so we arrive at a contradiction.
[1]

Definition 3.4. An *isometry* of a space is a linear transformation of that space that preserves length, such as rotations or translations. [1]

Definition 3.5. The *Haar measure* is a measure on the Borel σ-algebra that is invariant under translations. [8]

Theorem 3.6. If a measure-preserving transformation T on a measure space $(X, \mathcal{B}, \mu)$ is ergodic, then it is weak mixing if and only if 1 is the only eigenvalue of T.

We can begin by proving that if this transformation is weak mixing, then 1 is the only eigenvalue of T, which we can do by proving that if 1 is not the only eigenvalue of T, then the transformation is not weak mixing. We can consider an eigenfunction f of T along with its eigenvalue α. We can see that f is a factor map to a rotation $R_\alpha(z) = \alpha z$ on the unit circle, since the eigenfunction equation tells us that $f(T(x)) = \alpha(f(x)) = R_\alpha(f(x))$, so $f(T(x)) = R_\alpha(f(x))$. Since R_α is an isometry of the unit circle and if $\alpha \neq 1$, $f\mu$ is not a single atom, so X has non-trivial isometric factors and T is therefore not weak mixing.

Then, we need to prove that if 1 is the only eigenvalue of T, then T is weak mixing. We will begin by defining a group G and an element $g \in G$. Then, let $L_g : G \to G$ be the transformation $L_g(h) = gh$. If G is a compact group where the map is L_g then (G, L_g) is a group translation. Since the Haar measure is invariant under translations, we know that the Haar measure is automatically invariant, and if the Haar measure is ergodic then the translation is also ergodic.

[1]

4 Spectral Theorem

Definition 4.1. The *power set* $\mathcal{P}$ of a a set S is the set of all possible subsets of S.

Definition 4.2. S^1 indicates the unit circle in the complex plane.

I don't think I have the background information to give the proof of this theorem, so I will just give the statement of it.

Theorem 4.3. Let $U : H \to H$ be a unitary operator and $\mathbf{v} \in H$ a unit vector such that $\{U^n(\mathbf{v})\}_{n=-\infty}^{\infty} = H$. Then, there exists a probability measure $\mu_v \in \mathcal{P}(S^1)$ and a unitary operator $V : L^2(\mu) \to H$ such that $U = VMV^{-1}$, where $M : L^2(\mu) \to L^2(\mu)$ is defined by $M(f(z)) = zf(z)$. In addition, $V(1) = \mathbf{v}$. [1]

References

[1] Michael Hochman. Notes on ergodic theory, 2013.

[2] Eigenvalues.

[3] Compact space.

[4] Metric space.

[5] Cauchy sequence.

[6] Complete metric space.

[7] Hilbert space.

[8] Haar measure.