AN INTRODUCTION TO RATIONAL BILLIARDS

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1. Introduction

Billiards, with its rich history and diverse gameplay, encompasses a vast array of intriguing phenomena that extend far beyond the casual game played in pool halls.

In this exploration, we delve into the realm of rational billiards, focusing on the shapes of billiard tables and the trajectories of billiard balls. I will go through circular, elliptical, and triangular billiards, where we'll uncover the nature of periodic and non-periodic trajectories, examine the influence of geometric properties on ball movement, and discuss conjectures and theorems that shed light on the behavior of billiard systems.

Definition 1.1 (Billiards)**.** Billiards refers to the game involving the collision of billiard balls on a bounded table surface. The trajectories of the balls are influenced by the geometry of the table and the angles of impact.

Definition 1.2 (Shapes of Billiard Tables)**.** When we refer to shapes in the context of billiards, we are specifically discussing the geometric shape of the billiard table itself.

Note that we are looking at billiards from a strictly mathematical perspective, the goal being to find interesting results. We will be assuming that the ball does not experience any friction or physical forces other than the impact that puts it in motion.

Throughout our investigation, we aim to unravel the intricacies of rational billiards, offering insights into its mathematical nature.

2. Brief History

The history of rational billiards traces back to the ancient Greeks, who were among the first to study the geometric properties of billiard trajectories. However, the formal study of rational billiards as a mathematical discipline began to emerge in the late 19th and early 20th centuries.

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One of the pioneers in the field was mathematician Eugène Charles Catalan, who investigated the properties of billiard trajectories on tables with rational angles in the mid-1800s. He posed questions about the existence and classification of periodic trajectories on tables of different shapes, laying the groundwork for further exploration in the field.

In the early 20th century, the study of rational billiards gained momentum with the work of mathematicians such as George Birkhoff and his student Marston Morse. Birkhoff, known for his contributions to dynamical systems theory, made significant advancements in understanding the periodic and non-periodic behavior of billiard trajectories. His work laid the foundation for what would later become known as Birkhoff's Conjecture, which relates to the ergodic properties of billiard systems with rational angles.

Throughout the 20th century, mathematicians continued to explore various aspects of rational billiards, uncovering new periodic trajectories and studying the geometric properties of billiard tables. In particular, advancements in computer technology enabled researchers to investigate more complex billiard systems and analyze their dynamical behavior with greater precision.

In recent decades, the study of rational billiards has remained an active area of research, with mathematicians exploring connections to other fields such as number theory, geometry, and mathematical physics. New periodic trajectories continue to be discovered, and conjectures like Birkhoff's Conjecture continue to inspire further inquiry into the fascinating interplay between geometry and dynamics in billiard systems with rational angles.

3. Circular Billiards

Let us begin with an example of a simple shape to study the movements of the billiard balls, a circle. To begin to understand the circle we must first understand what it means to have a periodic trajectory.

Definition 3.1 (Periodic Trajectory)**.** We determine the trajectory of a billiard to periodic if it returns to its original position within a finite number of impacts.

We determine what the period is by counting the number of impacts.

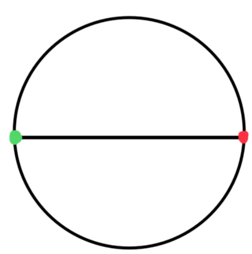

Figure 1. This is an example of a periodic trajectory of 3.

One more term we must know before understanding how to generate periodic trajectories is *Angle of Incidence*.

Definition 3.2 (Angle of Incidence)**.** The Angle of Incidence is the angle which an incident line makes with a perpendicular line to the wall at the point of contact.

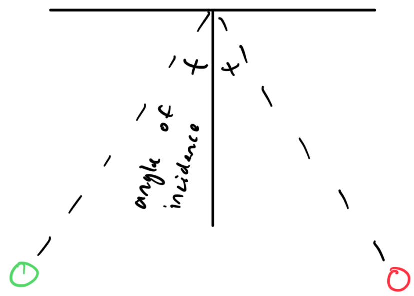

Figure 2. The labeled angle is the Angle of Incidence if the billiard ball starts at the green. Side-note: The other angle is equal and called the angle of reflection.

Now we can begin to understand how to generate periodic and non-periodic trajectories.

Generating Periodic and Non-Periodic Trajectories The first key observation is to find with what angle you must hit the ball such that the ball follows a periodic trajectory. Given that the circle is invariant under rotation, the position from where we hit the billiard ball does not matter, however, with what angle we do does.

Proposition 3.1. *If the ratio of the angle of incidence to π is rational then the trajectory is periodic. If the ratio is irrational, then the trajectory is non-periodic. We also notice that if the trajectory is periodic, the path of the ball must form a regular polygon. This is due to the fact that a circle is invariant under rotation. For example, if the period is 4, then we are essentially rotating the circle 4 times to come back to our starting point forming a square.*

Non-Periodic Trajectories One more interesting pattern to note is that the path of a non periodic trajectory is formed by individual "hits" of the billiard ball from one wall to another. If we look at these lines, we see that each ones is tangent to a circle and the entire path creates a circle within the bounds of the billiard table.

4. Elliptical Billiards

Exploring the dynamics of billiards on elliptical tables unveils fascinating phenomena and connections, providing insights into both theoretical and practical realms.

Elliptical billiards offer a natural extension from circular billiards, introducing more complex patterns and behaviors. When observing trajectories on an ellipse, it's notable that while some paths resemble smaller ellipses, trajectories passing through the foci of the ellipse are tangent to hyperbolas. Moreover, these trajectories eventually fill up the entire ellipse, a property absent in circular billiards.

Theorem 4.1 (Birkhoff-Poritsky Conjecture)**.** *A strictly convex C^2-smooth billiards table is locally integrable if and only if the table is an ellipse.*

This conjecture highlights the special integrability properties of elliptical billiards, suggesting that ellipses are unique among convex shapes in terms of local integrability. The theorem implies that elliptical billiards possess certain symmetries and properties that make their dynamics particularly tractable compared to other shapes.

Additionally, a notable feature specific to elliptical billiards is that a ball hit from one focus point will eventually reach the other focal point. This behavior has been practically exploited, as individuals have positioned themselves at the focal points of elliptical rooms to enhance their ability to hear conversations across the room.

One famous example where the phenomenon of sound focusing at the focal points of an elliptical room was utilized is the Whispering Gallery in St. Paul's Cathedral, London.

The Whispering Gallery is a circular balcony located around the interior of the cathedral's dome. Although the gallery itself is circular, the dome above it is elliptical in shape. Visitors to the gallery often discover that if they whisper close to the wall on one side of the gallery, the sound can be clearly heard by someone standing on the opposite side, even though the gallery spans a considerable distance.

This intriguing acoustic phenomenon occurs due to the elliptical shape of the dome. Sound waves originating at one focus of the ellipse are reflected off the walls and converge at the other focus. As a result, even a faint whisper can travel along the curved surface of the dome and be focused to the opposite side, allowing visitors to communicate across the gallery in a whispered conversation.

The Whispering Gallery serves as a real-world demonstration of the principle observed in elliptical billiards, where sound waves behave similarly to billiard balls bouncing off the walls of an elliptical room, converging at the focal points.

Further exploration of elliptical billiards reveals deep connections to various mathematical concepts and applications:

Geometric Properties Ellipses exhibit unique geometric properties that manifest in billiard dynamics. For instance, the constant sum of distances from any point on the ellipse to its two focal points plays a crucial role in determining trajectories. This property underlies the elegant behavior of billiard paths within elliptical boundaries.

Hamiltonian Dynamics The study of elliptical billiards often intersects with Hamiltonian dynamics, a branch of classical mechanics concerned with systems governed by Hamilton's equations. Elliptical billiards provide rich examples of integrable Hamiltonian systems, where trajectories can be precisely described using mathematical techniques such as action-angle variables.

5. Triangular Billiards

After discussing a simple shape like the circle, let's look at a more complex one such as the triangle.

We will start with a sub group of triangles and try to determine which ones have a periodic trajectory. Then we can slowly expand the types of triangles we take a look at.

Theorem 5.1. *There exists a periodic trajectory for all acute triangles. This is known as Fagnano's Trajectory.*

This trajectory allows us to construct a periodic path that a ball can take by doing the following:

Let there be an acute triangle ABC. Then if we draw in altitudes AD, BE, CF, and connect points D, E, and F, then we will have a periodic trajectory triangle ABC.

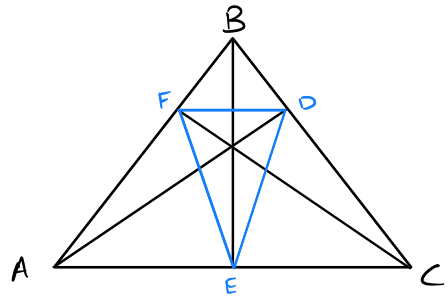

Figure 3. An example of a Fagnano Trajectory

Proof. We aim to demonstrate that the altitudes of triangle ABC divide the triangle formed by the bases D, E, and F of the altitudes into two equal parts. Let α, β, and γ denote the angles of triangle ABC.

Considering the right angles $\angle ADB$ and $\angle AEB$, we observe that points D and E lie on the semicircle with diameter AB. Consequently, we have $\angle ADE = \angle ABE$ (due to two inscribed angles subtending the same arc), and similarly, $\angle DEB = \angle DAB$. Since $\angle ABE = 90° - \alpha$ and $\angle DAB = 90° - \beta$, we can deduce the values of $\angle ADE$ and $\angle DEB$.

These calculations, in turn, lead to the derivation of the formulas for the six angles depicted in Figure 4. This concludes our proof. ■

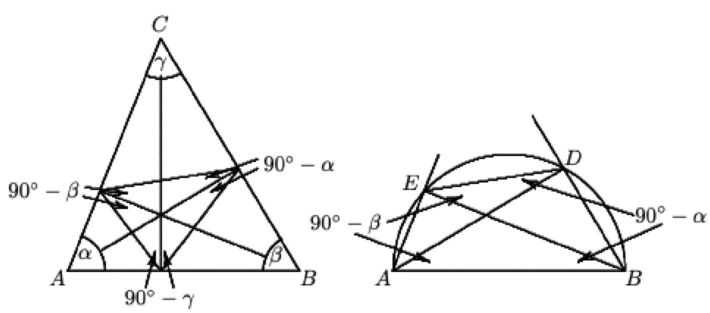

Figure 4

Now that we have shown that there exists a periodic trajectory, we can expand this thinking to figure out what other types of triangles have a periodic trajectory. The next logical choice would be right triangles.

Theorem 5.2. *There exists a periodic trajectory for all right triangles. The method to construct this is called shooting into the corner.*

To construct this trajectory, first take right triangle ABC. Then draw in altitude AE. If the billiard ball is chosen at some point next to the foot of the altitude and shot parallel to the altitude, it generates a periodic trajectory.

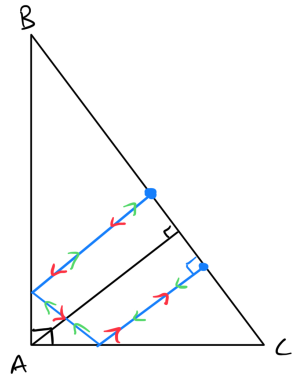

Figure 5. This is an example of shooting into the corner. The ball follows the path of the arrows.

From here, the next triangle we can look at are triangles with angles less than 100 degrees. Mathematician Richard Schwartz, figured this out with the aid of computer assistance in 2008 by breaking down the problem into multiple cases. In 2017, Jacob Garber, Boyan Marinov, Kenneth Moore, and George Tokarsky extended this to 112.3 degrees. Finally, in 1986, Masur proved the following theorem.

Theorem 5.3. *There exists a periodic trajectory for every triangle with rational angles.*

6. Sources

- https://youtu.be/GWzhNYkl6fA
- https://www.youtube.com/watch?v=nJWtSOxqyLo&t=59s&ab_channel=MathematicalCoinciden
- https://www.diva-portal.org/smash/get/diva2:650284/FULLTEXT01.pdf
- https://mathcircle.berkeley.edu/sites/default/files/archivedocs/2015/lecture/BMC_Advanced_11:3.pdf
- https://www.youtube.com/watch?v=zHugvX1kXug&ab_channel=AssociationforMathematicalR
- https://en.wikipedia.org/wiki/Fagnano%27s_problem