SPECTRAL TECHNIQUES

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1. Topic Summary

The study of Spectral Techniques in Ergodic Theory surrounds the implementation of eigenvalues in proving the ergodicity of various transformations. We first review fundamental definitions required for the study of these techniques, and then discuss important results.

2. Introduction to Spectral Techniques

We begin our paper with an introduction of a few definitions used to build the machinery we use later. The work of spectral techniques generally lies in the following L^2 space due to the nice properties of functions in said space and our ability to define an inner product (generalized dot product).

Definition 2.1. $L^2(X, \mu)$ space is the space of measurable functions from X to $\mathbb{R}$ that are square integrable; that is f such that $\int_X |f|^2 \, d\mu < \infty$. This space is Hilbert, meaning that it is a vector space equipped with an inner product $\langle f, g\rangle = \int_X f\overline{g}$ and corresponding norm $\sqrt{\langle f, f\rangle}$.

Furthermore, the composition of a function with a transformation T on X can be interpreted through a linear algebra sense by using inner products.

Definition 2.2. Let $U_T(f) := f \circ T$, where T is a measure preserving transformation. This is an isometry (function preserving inner product) on L^2, as $\langle U_T(f), U_T(g)\rangle = \int_X (f\overline{g}) \circ T = \int_X (f\overline{g}) = \langle f, g\rangle$.

The last definition we introduce to start off allows us to understand functions in terms of our linear algebra tools, which we will later apply to measurable functions.

Definition 2.3. If $f : X \to \mathbb{C}$; $f \in L^2$ satisfies $f \circ T = \lambda f$, f is called an eigenfunction, and λ is an eigenvalue. $H(T) := \{\lambda \in \mathbb{C} : f \circ T = \lambda f\}$ is the point spectrum, which consists of the eigenfunctions corresponding to T.

Note that when $\lambda = 1$, this is equivalent to f being T-invariant.

3. Ergodicity and Spectrality

A problem important to Ergodic Theory is determining whether two probability preserving spaces are equivalent measure theoretically. In class, we analyzed the uniqueness of measures, in particular invariant measures. Here, we will look at invariants on the weaker condition of spectral isomorphism as this condition is better understood.

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Definition 3.1. Two probability preserving spaces $(X, \mathcal{A}, \mu, T)$, $(Y, \mathcal{B}, \mu, S)$ (probability spaces equipped with measure preserving transformations T and S respectively) are said to be spectrally isomorphic if there exists a linear operator $W : L^2(X, \mu) \to L^2(Y, v)$ such that the following hold:

1. W is invertible
2. $\langle Wf, Wg\rangle = \langle f, g\rangle$ for all $f, g \in L^2(X, \mu)$
3. $WU_T = U_S W$

It is natural to show that spectral isomorphism is an equivalence relation, which can be seen through using the fact that inverses and compositions of linear operators satisfying our given conditions must also satisify all three of the conditions (albeit for different spaces). Thus, it also makes sense to define relations that hold for all/no spaces in a given equivalency class.

Definition 3.2. A property of a probability preserving space is a *spectral invariant* if it holds for all spaces that are spectrally isomorphic to it as well

For example, ergodicity and mixing are upheld under spectral isomorphism

Proof. In Chapter 6, Proposition 4.3, we proved that ergodicity is equivalent to $T-$invariant measurable functions being only those which are constant. Say T is ergodic. Then, if function g is S-invariant in $(Y, B, \check{;} S)$, its preimage g^* under W must be T-invariant as $g \circ S = U_S(g) = U_S W(g^*) = W(g^* \circ T)$ as desired, and reversing directions (due to the symmetry of spectral isomorphism) is sufficient.

Mixing can similarly be proven to be a spectral isomorphism, equivalent to ergodicity plus the following: $\lim_{n\to\infty}\langle f, U_T^n g\rangle = \langle f, 1\rangle\overline{\langle g, 1\rangle}$ (for $f, g \in L^2(X, \mu)$). Note that when we rewrite this without the inner products and in terms of simple functions, this is equivalent to our book definition. When we replace f and g with Wf and Wg, we get our desired result using property 2 of W. ■

Similarly, our aforementioned point spectrum $H(T)$ is a spectral invariant as well, which follows from W commuting with U_T and U_S depending on the direction. To apply this to the study of ergodicity, we need to show some important properties of the point spectrum:

Proposition 3.3. *Let $(X, \mathcal{A}, \mu, T)$ be an invariant probability measure space, where T is ergodic. Then:*
(1) $U_T f = \lambda f; f \in L^2\mu \to |\lambda| = 1$, $|f|$ constant with respect to U_T
(2) Eigenfunctions that correspond to different eigenvalues are orthogonal
(3) If f, g are both eigenfunctions for λ such $g \neq 0$ almost everywhere, then $f = cg$ for some constant c (a.e.)
(4) The eigenvalues form a subgroup of the unit circle

Proof.
(1) Note first that $|\lambda| = 1$ as T is measure preserving, which we showed alongside Definition 2.2. Thus, $|U_T^n f| = |f|$ is constant as desired.
(2) Let $U_T f = \lambda_1 f$, $U_T g = \lambda_2 g$. Note that $\langle f, g\rangle = \langle U_T f, U_T g\rangle = \lambda_1\overline{\lambda_2}\langle f, g\rangle$, which implies either $\lambda_1 = \lambda_2$ or $\langle f, g\rangle = 0$ as desired.
(3) If $|g| \neq 0$ almost everywhere, then note that $h = \frac{f}{g}$ is a T-invariant function and thus constant almost everywhere as well by ergodicity

(4) If $f \circ T = \lambda_1 f$ and $g \circ T = \lambda_2 f$, we have $\overline{f} \circ T = \overline{\lambda_1} f$, and that $(fg) \circ T = \lambda_1 \lambda_2 (fg)$, so all properties of a group are fulfilled (identity element and associativity are trivial in this example). ■

Note that we only used properties of ergodicity in proving property 3; the rest of these properties hold when T is measure preserving in general.

We can also view the types of spectra, which are also invariant under spectral isomorphism.

Definition 3.4. (Types of spectra (Def 3.4 of [2])) Given a probability preserving space $(X, \mathcal{A}, \mu, T)$, let V_d =span eigenfunctions. We say $(X, \mathcal{A}, \mu, T)$ has
1) Discrete Spectrum if $V_d = L^2$.
2) Continuous Spectrum if V_d is the set of constants.
3) Mixed Spectrum if V_d is neither of the above.

For example, we can show that irrational rotations have discrete spectrum; consider the rotation R_α on $\mathbb{R}/\mathbb{Z}$, and let $f_n(x) = nx$, so $f_n(\alpha x) = n\alpha + nx$. By Theorem 9.7 of [1], the set of f_n forms a basis of $L^2(\mathbb{R}/\mathbb{Z}, \mu)$, which is essentially equivalent to saying L^2 functions can be approximated by trigonometric polynomials (which follows from the Cesaro convergence of fourier series, though that is a topic for a different day), so R_α has a discrete spectrum. In fact, a similar argument can be made for general compact abelian groups and their Haar measure by considering the functions $f_a : G \to G$ sending x to ax (see [1] for more details).

Meanwhile, mixing transformations have continuous spectrum, as any non-constant eigenfunction f with eigenvalue λ has $\langle f, U_T^{n_k} \rangle \to ||f||_2^2 \neq (\int_X f)^2 = \int_X f \int_X f \circ T^{n_k}$, where the n_k are an increasing subsequence of positive integers such that $\lambda^{n_k} \to 1$ as $k \to \infty$. Note that the non-equality comes from applying Cauchy's Inequality and using the fact f is non-constant.

4. Spectral Techniques and Weak Mixing

We will now discuss the applications of spectral techniques in the study of weak mixing. An important theorem to start us off in our study is the following:

(3.1, modified from [3]) The following conditions are equivalent for a probability preserving system $(X, \mathcal{A}, \mu, T)$ on a Lebesgue space
1) Weak mixing
2) For every $A, B \in \mathcal{A}$, there exists $N \in \mathbb{N}$ of density 0 such that $\mu(A \cap T^{-n} B) \to \mu(A)\mu(B)$
3) $T \times T$ is ergodic (on the space $(X \times X, \mathcal{A} \times \mathcal{A}, \mu \times \mu, T \times T)$
4) Any eigenfunction $f \in L^2(X, \mu)$ is constant almost everywhere.

Proof. As an exercise in chapter 8, we proved that condition 1 implies condition 2 and vice versa (as weak Cesaro convergence is equivalent to convergence except on a set of measure 0). Furthermore, by then applying part b of exercise 14 in chapter 8, the first three conditions of this problem are equivalent.

Now, we show condition 3 implies condition 4. Suppose T is not weak mixing. Then, T has a non-constant eigenfunction f with eigenvalue λ. Then, $F(x, y) = f(x)\overline{f(y)}$ is $T \times T$ invariant, as $(T \times T)F(x, y) = \lambda\overline{\lambda} f(x)\overline{f(y)} = f(x)\overline{f(y)} = F(x, y)$. However, F must then be non-constant, which is a contradiction to the ergodicity of $T \times T$ as any T-invariant function is constant (Chapter 6, Prop 4.3 again). ■

Definition 4.1. An atom on a measure space $(X, \mathcal{A}, \mu)$ is a set A with non-zero measure such that for any measurable subset B of A, $\mu(B) = \mu(A)$ or 0. A measure space is said to be non-atomic if it has no atoms (such as $\mathbb{R}$ with the Lebesgue measures)

Definition 4.2. The spectral measure of $f \in L^2 \setminus 0$ is the unique measure v_f on S^1 (the unit circle) such $\langle f \circ T^n, f\rangle = \int_{S^1} z^n \, dv_f$ for $n \in \mathbb{Z}$.

Proposition 4.3. *If T is weak mixing on a Lebesgue probability space, then all spectral measures of $f \in L^2$ such that $\int f \neq 0$ are non-atomic.*

Proof. See Proposition 3.3 of [3] ■

Why is this important? The following theorem shows the significance of non-atomicity:

Theorem 4.4. *Isomorphism Theorem - A.2 of [3] Every non-atomic standard probability space is isomorphic to the unit interval equipped with the Lebesgue measure.*

We now work on a theorem whose proof revolves around tools been built up throughout this paper:

Proposition 4.5. *(3.4) in [2] If T is weak-mixing, then for all $f, g \in L^2$, $\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} |\int g(f \circ T^n) d\mu - (\int f d\mu)(\int g d\mu) = 0$*

We first begin with the following lemma:

Lemma 4.6. *If T is weak mixing, then for every $f \in L^2$, we have $\frac{1}{n}\sum_{k=0}^{n-1} |\int_X f \cdot (f \circ T^n)\, d\mu - (\int_X f \, d\mu)^2| \to 0$ as $n \to \infty$. This is equivalent to saying Proposition 4.3 holds when $f = g$.*

Proof. We only need to consider the case where $\int f \neq 0$ by shifting it by a constant. Now, letting v_f be the spectral measure, we find that $\frac{1}{N}\sum_{k=0}^{N-1} |\int_X f \cdot (f \circ T^n)\, d\mu|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |\langle U_T^n f, f\rangle|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |\int_{S_1} z^n \, dv_f(z)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} (\int_{S_1} z^n \, dv_f(z))\overline{(\int_{S_1} z^n \, dv_f(z))} = \frac{1}{N}\sum_{k=0}^{N-1} (\int_{S_1}\int_{S_1} z^n \overline{w^n}\, dv_f(z) v_f(w) = \int_{S_1}\int_{S_1} \frac{1}{N}\sum_{k=0}^{N-1} z^n \overline{w^n}$. Each term in the integrand tends to 0 except when $w = z$, in which case it is 1. However, the set where $w = z$ has measure 0 wrt to $S_1 \times S_1$, so when we sum our integral overall it approaches 0 as $N \to \infty$ (however, this requires some precision since we use the fact v_f is non-atomic in doing so). ■

We now conclude the proof of our proposition

Proof. First, assume T is invertible, and thus U_T is as well. Fix $f \in L^2$ and let $S(f) := \text{span}\{U_T^k f : k \in \mathbb{Z}\}$. Note that as L^2 is a vector space, we can write it as $L^2 = S(f) + (S(f) + c_1)^{\perp} + c_2$, where c_i denotes a constant function (we have to be careful to include constants in this example). By our lemma, any $g \in S(f)$ instantly satisfies our claim, as it can be expressed as a linear combination of $U_T^k f$ which satisfy our result as well by T being measure preserving. Similarly, constants also satisfy our proposition, so it is sufficient to show that functions g orthogonal to $S(f) + c_1$ satisfy the proposition. However, this follows from $\langle g, f \circ T^n\rangle$ approaching 0 (which is equivalent to our first integral).

If T is NOT invertible, we can find an extension $\overline{T}$ that is invertible on a larger probability preserving space, and thus satisfies our proposition. Using some manipulations (see 3.4 of [3]), T must then also satisfy the proposition. ■

5. Sources Implemented

[1] Mirzakhani, Maryam, and Tony Feng. Introduction to Ergodic Theory. 2014. https://math.berkeley.edu/~fengt/ergodic_theory.pdf

[2] Rubinstein-Salzedo, Simon. Ergodic Theory.

[3] Sarig, Omri. Omri Sarig Lecture Notes on Ergodic Theory. 2023. https://www.ime.usp.br/~sylvain/ErgodicSpectral.pdf

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