THE HAAR MEASURE ON A LOCALLY COMPACT GROUP

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ABSTRACT. In this paper we introduce and define the Haar measure on a locally compact group G. We give some properties of integration with respect to a Haar measure and then prove that on any locally compact group there exists an essentially unique Haar measure. Finally, we give several examples of this measure on different locally compact groups.

1. Introduction and Motivation

One of the most important properties of the Lebesgue measure λ on $\mathbb{R}^n$ is that of translation invariance:

$$\lambda(x + A) = \lambda(A)$$

for all $x \in \mathbb{R}^n$ and measurable A. One way to think about this property is that it says that λ is uniform; it assigns the same weight to each point. This is because the sets A and $x + A$ are essentially the same set, just at different points in space. Clearly, a uniform measure should be one for which the measures of these sets are the same.

The Haar measure is an extension of this idea to a very general setting, namely that of topological groups. A topological group is a set with both a topology and group structure, along with a couple compatibility conditions connecting the two, which will be stated in Section 3. In general a topological group G may not be commutative, so we have two different kinds of Haar measures, ones which satisfy translation invariance on the left and ones which satisfy translation invariance on the right. More precisely, we say that μ is a left Haar measure if it satisfies some "niceness" conditions and

$$\mu(gA) = \mu(A)$$

for all $g \in G$ and measurable A, and we make an analogous definition for right Haar measures.

Unlike with $\mathbb{R}^n$, it may not be immediately obvious why translation invariance is a good way to formalize the idea of a uniform measure in this general setting. However, this does make sense. Given a topological group G, Cayley's Theorem tells us that as a group, G is isomorphic to a group of symmetries, namely the translations $T_g : G \to G$ sending x to gx. The compatibility conditions for a topological group ensure that these translations preserve the topological structure of G. We can thus think of the group structure on G as defining a group of symmetries of G as a topological space. A Haar measure can then be thought of naturally as a measure which respects these symmetries, i.e. a measure obtained by assuming these symmetries to be measure-preserving.

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As we will show in Section 5, it turns out that on any sufficiently nice (locally compact and Hausdorff) topological group, there always exists a left Haar measure. Further, this measure is unique up to multiplication by a positive constant.

2. Topological Preliminaries

In this section we give a quick list of definitions and theorems from point-set topology which we will need to properly discuss the Haar measure. For proofs of the theorems, see [6].

Definition 1. A *topological space* is an ordered pair (X, τ), where X is a set and τ is a collection of subsets of X satisfying the following properties:

(i) $\varnothing \in \tau$ and $X \in \tau$
(ii) If $\{U_\alpha\}$ is any collection of elements of τ, then $\bigcup_\alpha U_\alpha \in \tau$.
(iii) If $U_i \in \tau$ for $i = 1, 2, \ldots, n$, then $\cap_{i=1}^n U_i \in \tau$.

If the reference to the topology is clear, then we often refer to X alone as a topological space.

Definition 2. If (X, τ) is a topological space, then the elements of τ are called *open* sets. A set $E \subset X$ is called *closed* if its complement E^c is open.

It is not difficult to show that the intersection of an arbitrary family of closed sets is closed and the finite union of closed sets is closed. In fact, one can equivalently define a topology by specifying a set of closed sets which satisfy these properties, along with the property that $\varnothing$ and the whole set are closed.

Example 3. If X is any set, the *discrete topology* is defined to be the topology such that every subset of X is open. The *indiscrete* topology is the topology containing only the sets X and $\varnothing$.

Definition 4. If $A \subset X$ is a subset of a topological space, then the *closure* of A, denoted $\overline{A}$ is defined to be the intersection of all closed sets containing A. Similarly, the *interior* of A, denoted A° is defined as the union of all open sets contained in A.

By definition, an arbitrary union of open sets is open, so the interior of A can be thought of as the largest open set contained in A. Similarly, an arbitrary intersection of closed sets is closed, so it follows that the closure of a set A is closed and can thus be interpreted as the smallest closed set containing A.

Definition 5. If X is a topological space and $x \in X$, then an *open neighborhood* of x is an open set U with $x \in U$.

Definition 6. If X and Y are topological spaces and $f : X \to Y$ is a function, then f is said to be *continuous* if $f^{-1}(U)$ is an open set for all open sets $U \subset Y$.

Theorem 7. *If X, Y, and Z are topological spaces, $f : X \to Y$ is continuous, and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is continuous.*

Definition 8. A function $f : X \to Y$ between two topological spaces is said to be a *homeomorphism* if f is continuous and there exists a continuous function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, where id_X and id_Y are the identity functions on X and Y, respectively.

If there exists a homeomorphism between two topological spaces X and Y, then X and Y are said to be homeomorphic. It is not difficult to verify that the notion of homeomorphic spaces is an equivalence relation.

Definition 9. Let f be a complex-valued continuous function on a topological space X. The *support* of f is defined to be

$$\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}},$$

i.e. it is the closure of the set of points at which f is nonzero.

Definition 10. A topological space X is said to be *Hausdorff* if for any two distinct points $x, y \in X$, we can find disjoint open sets U and V containing x and y, respectively.

Definition 11. A subset K of a topological space X is said to be *compact* if every open cover of K has a finite subcover. More precisely, we require that if $\{U_\alpha\}$ is any collection of open sets whose union contains K, there there exists a finite number of sets in $\{U_\alpha\}$ whose union contains K. If X itself is compact, then X is said to be a *compact space.*

Theorem 12. *If X is a topological space and K_1, K_2, ..., K_n are compact, then $\bigcup_{i=1}^n K_i$ is compact.*

Theorem 13. *Let X and Y be topological spaces and let $f : X \to Y$ be continuous. Then, if $K \subset X$ is compact, $f(K) \subset Y$ is compact.*

Theorem 14. *If K is a compact subset of a Hausdorff space, then K is closed.*

Theorem 15. *If K is compact and $A \subset K$ is closed, then A is compact.*

Theorem 16. *Let X be a Hausdorff space, and let K_1 and K_2 be compact subspaces of X. Then, if $K_1 \cap K_2 = \varnothing$, we can find open sets $U_1 \supset K_1$ and $U_2 \supset K_2$ such that $U_1 \cap U_2 = \varnothing$.*

Definition 17. A topological space X is said to be *locally compact* if every point in X has a neighborhood which is contained in a compact set.

We note that if a topological space X is Hausdorff, then the previous definition is equivalent to saying each point in X has an open neighborhood with compact closure, by Theorems 14 and 15.

Definition 18. Let X be a set and let $\mathcal{A}$ be a collection of subsets of X. Then $\mathcal{A}$ is said to have the *finite intersection property* if for all $A_1, A_2, \ldots, A_n \in \mathcal{A}$, $\bigcap_{i=1}^n A_i \neq \varnothing$.

Theorem 19. *Let X be a topological space. Then X is compact if and only if for any collection of closed sets $\mathcal{F}$ having the finite intersection property, the intersection $\bigcap_{F \in \mathcal{F}} F$ is nonempty.*

Definition 20. Let X_i be a topological space for all $i \in I$, then the set $\prod_{i \in I} X_i$ inherits a natural topology, called the *product topology*, which is defined to be the smallest topology containing all sets of the form $\prod_{i \in I} U_i$, where each $U_i \subset X_i$ is open and $U_i = X_i$ for all but finitely many values of i.

Theorem 21. *Let $X = \prod_{i \in I} X_i$, where X_i is a topological space for all $i \in I$, and endow X with the product topology. Then, each projection map $\pi_i : X \to X_i$ is continuous.*

Theorem 22. *Let X be a topological space, let Y_i be a topological space for all $i \in I$, and let $Y = \prod_{i \in I} Y_i$ be the product space. Then, a function $f : X \to Y$ is continuous if and only if $\pi_i \circ f : X \to Y_i$ is continuous for all $i \in I$, where $\pi_i : Y \to Y_i$ is the projection map.*

Theorem 23 (Heine–Borel)**.** *A set $A \subset \mathbb{R}^n$ is compact in the product topology if and only if A is closed and bounded.*

Theorem 24 (Tychonoff's Theorem)**.** *If X_i is a compact topological space for all $i \in I$, then $\prod_{i \in I} X_i$ is compact in the product topology.*[1]

Definition 25. If X is a topological space and $S \subset X$, then S inherits a natural topology called the *subspace topology* by defining the open sets of S to be sets of the form $U \cap S$, where U is open in X.

We will also prove a lemma which we will need later.

Lemma 26. *Let X be a Hausdorff space, let K be a compact subset of X, and let U_1 and U_2 be open sets such that $K \subset U_1 \cup U_2$. Then there exist compact sets $K_1 \subset U_1$ and $K_2 \subset U_2$ such that $K = K_1 \cup K_2$.*

Proof. Let $L_1 = K - U_1$ and $L_2 = K - U_2$. K is closed by Theorem 14, so L_1 and L_2 are closed. Since L_1 and L_2 are closed subsets of the compact set K, L_1 and L_2 are compact by Theorem 15. Since L_1 and L_2 are disjoint compact sets, Theorem 16 says we can find disjoint open sets $V_1 \supset L_1$ and $V_2 \supset L_2$. Let $K_1 = K - V_1$ and $K_2 = K - V_2$. K_1 and K_2 are compact because they are closed subsets of the compact set K. Also, since V_1 and V_2 are disjoint, $K_1 \cup K_2 = K$. Finally, we have

$$K_1 = K - V_1 \subset K - L_1 = K \cap L_1^c = K \cap (K \cap U_1^c)^c = K \cap (K^c \cup U_1) = K \cap U_1 \subset U_1,$$

and similarly $K_2 \subset U_2$. □

3. Topological Groups and the Haar Measure

A topological group is essentially just a set G with both a topology and a group structure. However, this alone is uninteresting; in general we couldn't do much more than study the topological structure and group structure of such a set separately. What we really want is a set which has a topology and a group structure which interacts nicely with this topology. This is encapsulated in the following definition.

Definition 27. A *topological group* is a topological space G which is also a group such that the map $(x, y) \to xy$ from $G \times G$ to G and the inversion map $x \to x^{-1}$ from G to G are both continuous.

An important fact that follows directly from this definition is that for all $g \in G$, the translation map $T_g : G \to G$ sending x to gx is a homeomorphism, since it is continuous and has continuous inverse $T_{g^{-1}}$.

Example 28. Any finite group G with the discrete topology is a topological group because all functions from a discrete space are continuous.

Example 29. $\mathbb{R}^n$ with its standard topology and additive group structure is a topological group because the functions $(x, y) \to x + y$ and $x \to -x$ are continuous.

[1] The axiom of choice is necessary to prove this result in general. However, the case of only finitely many spaces X_i is true even without assuming the axiom of choice.

Example 30. The multiplicative group of positive real numbers $\mathbb{R}_{>0}$ is a topological group with the subspace topology inherited from $\mathbb{R}$, because the functions $(x, y) \to xy$ and $x \to \frac{1}{x}$ are continuous on $\mathbb{R}_{>0}$.

The definition of a topological space is very general, which allows for many types of pathological spaces. For this reason, we often need to add several assumptions when working with general topological spaces. For the Haar measure, we need our topological space to be both Hausdorff and locally compact, both of which are properties satisfied by basically all sufficiently nice spaces. To simplify terminology, we make the following definition.

Definition 31. A *locally compact group* is a topological group for which the underlying topology is locally compact and Hausdorff.

We are now ready to give the full definition of a Haar measure.

Definition 32. A *left Haar measure* is a measure μ, not identically 0, defined on the Borel sets of a locally compact group G and satisfying the following four properties:

(1) μ is left-translation-invariant: $\mu(gE) = \mu(E)$ for Borel sets E and all $g \in G$.
(2) $\mu(K) < \infty$ for all compact sets K.
(3) μ is outer regular: for all Borel sets E,
$$\mu(E) = \inf\{\mu(U) \mid E \subset U, U \text{ open}\}.$$
(4) μ is inner regular: for all open sets U,
$$\mu(U) = \sup\{\mu(K) \mid K \subset U, K \text{ compact}\}.$$

A measure defined on the Borel sets of a topological space which satisfies properties (2), (3), and (4) is called a regular Borel measure. This is essentially a continuity property, which says that sets can be approximated from the outside with open sets and approximated from the inside with compact sets. A right Haar measure is defined analogously. Clearly, left and right Haar measures coincide when the locally compact group G is abelian. In fact, it turns out that left and right Haar measure also coincide if the group G is compact, although this is not obvious. See [3] for example.

The most important result about Haar measures is that every locally compact group has a left Haar measure, and this measure is unique up to multiplication by a positive constant. We will prove this in Section 5.

Example 33. If G is any finite group with the discrete topology then the Haar measure on G is, up to multiplication by a positive constant, the counting measure:
$$\mu(A) = |A|$$
for all $A \subset X$.

Example 34. The Haar measure on $\mathbb{R}^n$ for which the cube $[0, 1]^n$ has measure 1 is the restriction of the Lebesgue measure to Borel sets.

Example 35. The Haar measure on the multiplicative group of positive real numbers $\mathbb{R}_{>0}$ is the measure
$$\mu(A) = \int_A \frac{1}{t}\, dt$$

Translation invariance of this measure by some $x \in \mathbb{R}_{>0}$ follows from the substitution $t = xu$:

$$\mu(xA) = \int_{xA} \frac{1}{t}\, dt = \int_A \frac{1}{xu} \cdot x\, du = \int_A \frac{1}{u}\, du = \mu(A).$$

For more examples see Section 6. We finish this section by proving a couple results about topological groups which we will need to prove the existence and uniqueness of the Haar measure.

Theorem 36. *Let G be a topological group and let U be an open set containing the identity element $e \in G$. Then there exists an open set V containing the identity such that $VV \subset U$.*

Proof. Let $f : G \times G \to G$ denote the group operation map, i.e. $f(x, y) = xy$. Since U is open and f is continuous,

$$f^{-1}(U) = \{(x, y) \in G \times G \mid xy \in U\}.$$

Note that $(e, e) \in f^{-1}(U)$ because $e \in U$. By definition of the product topology, we can find open sets $V_1, V_2 \subset G$ such that $V_1 \times V_2 \subset f^{-1}(U)$ and $(e, e) \in V_1 \times V_2$, meaning $e \in V_1$ and $e \in V_2$. It follows that $V = V_1 \cap V_2$ is an open set containing e. Finally, we have $VV \subset U$ because for all $v_1, v_2 \in V$, $(v_1, v_2) \in V_1 \times V_2 \subset f^{-1}(U)$, so $v_1 v_2 \in U$. □

Theorem 37. *Let G be a topological group, let $K \subset G$ be compact, and let $U \supset K$ be open. Then there exists an open set V containing the identity of G such that $KV \subset U$.*

Proof. For each $x \in K$, let $W_x = x^{-1}U$, which is an open set containing the identity. By Theorem 36, we can choose an open set V_x containing the identity such that $V_x V_x \subset W_x$. It follows that the collection of sets $\{xV_x \mid x \in K\}$ is an open cover of K. Since K is compact, we can find finitely many points $x_1, x_2, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n x_i V_{x_i}$. Finally, define $V = \bigcap_{i=1}^n V_{x_i}$, which we claim satisfies the desired property. Indeed, for all $x \in K$, we can find some x_i such that $x \in x_i V_{x_i}$ and thus

$$xV \subset x_i V_{x_i} V \subset x_i V_{x_i} V_{x_i} \subset x_i W_{x_i} = U.$$

It follows that $KV \subset U$. □

4. The Haar Integral

If μ is a Haar measure on a locally compact group G, then using the general theory of integration with respect to a measure, we can define an integral of any Borel measurable function on G with respect to μ. This integral is called the Haar integral. The translation invariance of the Haar measure gives us a very useful property of this integral:

Theorem 38. *If μ is a left Haar measure on a locally compact group G then for all functions $f \in L^1(\mu)$ and $x \in X$,*

$$\int_G f(x)\, d\mu(x) = \int_G f(gx)\, d\mu(x).$$

Proof. First consider the case when $f = \chi_E$ is the characteristic function for some Borel set E. We then have

$$\int_G \chi_E(x)\, d\mu(x) = \mu(E) = \mu(g^{-1}E) = \int_G \chi_{g^{-1}E}(x)\, d\mu(x) = \int_G \chi_E(gx)\, d\mu(x),$$

since $\chi_E(gx) = \chi_{g^{-1}E}(x)$. The theorem then immediately follows for all measurable simple functions on G.

Now, if $f : G \to [0, \infty]$ is any positive Borel measurable function, then we can find a sequence of simple measurable functions $s_i(x)$ such that $0 \le s_1(x) \le s_2(x) \le \cdots \le f(x)$ and $s_n(x) \to f(x)$ as $n \to \infty$ for all $x \in G$. It follows that the functions $s_i(gx)$ are simple measurable functions which satisfy $0 \le s_1(gx) \le s_2(gx) \le \cdots \le f(gx)$ and $s_n(gx) \to f(gx)$ as $n \to \infty$ for all $x \in G$. Lebesgue's monotone convergence theorem applied to each of these sequences then gives

$$\int_G f(x)\, d\mu(x) = \lim_{n\to\infty} \int_G s_n(x)\, d\mu = \lim_{n\to\infty} \int_G s_n(gx)\, d\mu = \int_G f(gx)\, d\mu(x).$$

The theorem then follows for any complex function $f \in L^1(\mu)$ by writing $f = u^+ - u^- + iv^+ - iv^-$, where u^+, u^-, v^+, and v^- are all positive Borel measurable functions. □

Interestingly, it turns out that on a locally compact group G, any function which has some basic integral properties (positive, linear, and defined on the set of all continuous functions with compact support) and satisfies the above property, must be an integral with respect to a Haar measure. To show this, we first need some definitions.

Notation 39. The collection of all continuous complex functions with compact support on a topological space X is denoted $C_c(X)$.

It is not difficult to verify that $C_c(X)$ is a vector space for any topological space X.

Definition 40. A *positive linear functional* on $C_c(X)$ is a linear map $\Lambda : C_c(x) \to \mathbb{C}$ such that $\Lambda(f)$ is real and nonnegative for any nonnegative $f \in C_c(X)$.

Definition 41. A *left-invariant integral* on a locally compact group G is a positive linear functional $\Lambda : C_c(G) \to \mathbb{C}$ with the additional property that Λ is invariant under the group operation:

$$\Lambda(f(x)) = \Lambda(f(gx))$$

for all $g \in G$ and $f \in C_c(G)$.

Note that the property that Λ is invariant under the group operation is exactly what Theorem 38 says for integration with respect to a Haar measure. We will also need the following well-known theorems.

Theorem 42 (Riesz Representation Theorem)**.** *Let X be a locally compact Hausdorff topological space, and let Λ be a positive linear functional on $C_c(X)$. Then there exists a unique regular Borel measure μ such that*

$$\Lambda(f) = \int_X f\, d\mu$$

for all $f \in C_c(X)$.

Theorem 43 (Urysohn's Lemma)**.** *Suppose X is a locally compact Hausdorff space, V is open in X, $K \subset V$, and K is compact. Then there exists an $f \in C_c(X)$ such that $0 \leq f(x) \leq 1$ for all $x \in X$, $f(x) = 1$ for all $x \in K$, and the support of f is contained in V.*

For proofs of these, see [7]. We can now show the following:

Theorem 44. *Let G be a locally compact group and suppose $\Lambda : C_c(G) \to \mathbb{C}$ is a left-invariant integral. Then there exists a unique Haar measure μ such that*

$$\Lambda(f) = \int_G f \, d\mu$$

for all $f \in C_c(G)$.

Proof. By Theorem 42, we know there exists a unique measure μ such that $\Lambda(f) = \int_G f \, d\mu$ for all $f \in C_c(G)$. Properties (b), (c), and (d) of Theorem 42 show that μ restricted to Borel sets satisfies properties (2), (3), and (4) of a left Haar measure. We thus only need to show that μ is left translation invariant. We first show this for open sets. Let $V \subset G$ be open, let $g \in G$ be arbitrary, and let $\epsilon > 0$ be given. By the regularity of μ, we can find a compact set $K \subset V$ such that

$$\mu(V) - \mu(K) < \epsilon.$$

Now, by Urysohn's lemma, we can find some $f \in C_c(G)$ such that

$$\chi_K(x) \leq f(x) \leq \chi_V(x)$$

for all $x \in X$. Replacing x with $g^{-1}x$, this gives

$$\chi_{gK}(x) \leq f(g^{-1}x) \leq \chi_{gV}(x)$$

Integrating both these inequalities over G, we find

$$\mu(K) \leq \int_G f(x) \, d\mu(x) = \Lambda(f) \leq \mu(V).$$

$$\mu(gK) \leq \int_G f(g^{-1}x) \, d\mu(x) = \Lambda(f) \leq \mu(gV).$$

It then follows that

$$\mu(gV) \geq \Lambda(f) \geq \mu(K) > \mu(V) - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this implies $\mu(gV) \geq \mu(V)$. Now, by the same argument but replacing g with g^{-1} and V with gV, we get the reverse inequality, $\mu(V) \geq \mu(gV)$. Thus,

$$\mu(gV) = \mu(V),$$

so μ is left translation invariant on open sets. Now let $E \subset X$ be any Borel set. By the regularity of μ, we have

$$\mu(E) = \inf\{\mu(V) \mid E \subset V, V \text{ open}\}$$

and

$$\mu(gE) = \inf\{\mu(V) \mid gE \subset V, V \text{ open}\}.$$

Since the translation map $x \to gx$ is a homeomorphism, it follows that V is an open set containing E if and only if gV is an open set containing gE. Thus, using the fact that $\mu(gV) = \mu(V)$ for all open sets V,

$$\begin{aligned}\mu(E) &= \inf\{\mu(V) \mid E \subset V, V \text{ open}\} \\ &= \inf\{\mu(gV) \mid E \subset V, V \text{ open}\} \\ &= \inf\{\mu(V) \mid gE \subset V, V \text{ open}\} \\ &= \mu(gE),\end{aligned}$$

as desired. □

5. Existence and Uniqueness of the Haar Measure

The goal of this section is to prove the existence and uniqueness of the Haar measure on any locally compact group G. The proof here is based on [4], [5], and [2], and uses the axiom of choice in the form of Tychonoff's theorem. However, it is possible to prove without this, as in [1], although this is more difficult.

Theorem 45. *On any locally compact group G there exists at least one left Haar measure.*

Proof. We start with a definition.

Definition 46. If $K \subset G$ is compact and $V \subset G$ has nonempty interior V°, then we define the quantity $(K : V)$ to be the smallest nonnegative integer n such that there exist $g_1, g_2, \dots, g_n \in G$ satisfying

$$K \subset \bigcup_{i=1}^{n} g_i V^\circ.$$

In other words, $(K : V)$ is the smallest number of translates of V° that cover V. This quantity is always well defined and finite, since for all such K and V, the set $\{gV^\circ \mid g \in G\}$ is an open cover of G, and hence K. By compactness of K there exists a finite subcover, implying $(K : V)$ is finite. The existence of a minimum number of translates of V° that cover V then follows since the set of natural numbers is well-ordered.

The idea is that for an open set U, $(K : U)$ measures the size of K with the set U, and we can get more accurate values of this size by taking U to be smaller. Of course, if we simply take a limit as U gets smaller, the quantity $(K : U)$ might just go to infinity, so we first need to normalize. Thus, choose some compact set K_0 with nonempty interior. Then, $(K_0 : U) > 0$ for all open sets U. For all open sets U, we can now define a function h_U on the compact sets $K \subset G$ by

$$h_U(K) = \frac{(K : U)}{(K_0 : U)}.$$

$h_U(K)$ is essentially the size of K relative to K_0, measured using the set U. The idea is to take a sort of limit as U gets small, although it's not obvious how to do this.

First, we restrict our attention to the collection of open sets U which contain the identity of G. Let $\mathcal{U}$ denote this collection of sets. We don't really lose anything by doing this because for any open set U, we can find some $g \in G$ such that $gU \in \mathcal{U}$ and then $h_{gU}(K) = h_U(K)$ for all K. The benefit of restricting our attention to

$\mathcal{U}$ is that we now have a directed family of subsets: for all $U_1, U_2 \in \mathcal{U}$ we have $U_1 \cap U_2 \in \mathcal{U}$ and $U_1 \cap U_2$ is smaller than U_1 and U_2.

Before we proceed, it will be helpful to prove a few properties of h_U.

Lemma 47. *Let $U \in \mathcal{U}$, let K, K_1, K_2 be compact, and let $g \in G$. Then,*

(1) $0 \leq h_U(K) \leq (K : K_0)$.
(2) $h_U(gK) = h_U(K)$.
(3) *If $K_1 \subset K_2$, then $h_U(K_1) \leq h_U(K_2)$.*
(4) $h_U(K_1 \cup K_2) \leq h_U(K_1) + h_U(K_2)$.
(5) *If $K_1U^{-1} \cap K_2U^{-1} = \varnothing$ then $h_U(K_1 \cup K_2) = h_U(K_1) + h_U(K_2)$.*

Proof.

(1) The inequality $0 \leq h_U(K)$ is obvious. Now, let $m = (K : K_0)$ and $n = (K_0 : U)$. Then choose $g_1, g_2, \ldots, g_m \in G$ such that $K \subset \bigcup_{i=1}^m g_i K_0^\circ$ and choose $h_1, h_2, \ldots, h_n \in G$ such that $K_0 \subset \bigcup_{i=1}^n h_i U$. We then have

$$K \subset \bigcup_{i=1}^{m} g_i K_0^\circ \subset \bigcup_{i=1}^{m} \bigcup_{i=1}^{n} g_i h_i U$$

and thus

$$(K : U) \leq mn$$

by definition. Dividing this by $m \neq 0$, we obtain

$$h_U(K) \leq (K : K_0).$$

(2) We note that given a cover of K by translates of U, we can translate each set in this cover by g to get a cover of gK by the same number of translates of U. Conversely, given a cover of gK by translates of U, we can translate each set in this cover by g^{-1} to get a cover of K by the same number of translates. It follows that $(K : U) = (gK : U)$ and thus $h_U(gK) = h_U(K)$.

(3) Suppose that $K_1 \subset K_2$. Then, for all $U \in \mathcal{U}$, any covering of K_2 by translates of U is a covering of K_1 by translates of U, so $(K_1 : U) \leq (K_2 : U)$. Dividing by $(K_0 : U)$, we get $h_U(K_1) \leq h_U(K_2)$ for all $U \in \mathcal{U}$.

(4) Take an open cover of K_1 with $(K_1 : U)$ translates of U and take an open cover of K_2 with $(K_2 : U)$ translates of U. The union of these covers is then an open cover of $K_1 \cup K_2$ with $(K_1 : U) + (K_2 : U)$ translates of U. Thus,

$$(K_1 \cup K_2 : U) \leq (K_1 : U) + (K_2 : U)$$

which implies $h_U(K_1 \cup K_2) \leq h_U(K_1) + h_U(K_2)$.

(5) Suppose that K_1 and K_2 satisfy $K_1U^{-1} \cap K_2U^{-1} = \varnothing$. The condition $K_1U^{-1} \cap K_2U^{-1} = \varnothing$ is the same as saying that no translate of U intersects both K_1 and K_2. Now, choose a cover of $K_1 \cup K_2$ by $(K_1 \cup K_2 : U)$ translates of U. By the condition $K_1U^{-1} \cap K_2U^{-1} = \varnothing$, each translate intersects exactly one of K_1 and K_2. Let g_1U, g_2U, $\ldots$, g_nU be the translates that intersect K_1 and h_1U, h_2U, $\ldots$, h_mU be the translates that intersect K_2, with $m + n = (K_1 \cup K_2 : U)$. It follows that the sets g_1U, g_2U, $\ldots$, g_nU cover K_1 and the sets h_1U, h_2U, $\ldots$, h_mU cover K_2, so $(K_1 : U) \leq n$ and $(K_2 : U) \leq m$. We thus have

$$(K_1 : U) + (K_2 : U) \leq n + m = (K_1 \cup K_2 : U),$$

which gives $h_U(K_1) + h_U(K_2) \leq h_U(K_1 \cup K_2)$. We showed the opposite inequality in the previous part, so it follows that $h_U(K_1) + h_U(K_2) = h_U(K_1 \cup K_2)$.

□

We now define a new function h by taking a sort of limit of h_U as U gets smaller. Let $\mathcal{K}$ denote the set of compact subsets of G. Since by Lemma 47, $0 \leq h_U(K) \leq (K : K_0)$, we can consider each function h_U for $U \in \mathcal{U}$ as a point in the set $X = \prod_{K \in \mathcal{K}} [0, (K : K_0)]$. Endowed with the product topology, X is compact by Tychonoff's Theorem (Theorem 24). For each $V \in \mathcal{U}$, define the set $C(V) \subset X$ by

$$C(V) = \overline{\{h_U \mid U \in \mathcal{U}, U \subset V\}}.$$

i.e. the set $C(V)$ is the set of functions h_U for U smaller than V (along with all limits of sequences of such functions, since we are taking the closure). We want a measure h which is in some sense the limit of the measures h_U as U gets small, so we better have $h \in C(V)$ for all V, i.e. $h \in \bigcap_{V \in \mathcal{U}} C(V)$. To show that such a measure actually exists, meaning that this intersection is nonempty, we will show that the collection of sets $\{C(V)\}_{V \in \mathcal{U}}$ has the finite intersection property and apply Theorem 19. Thus, suppose $V_1, V_2, \dots, V_n \in \mathcal{U}$. Then, $\bigcap_{i=1}^n V_i \in \mathcal{U}$ so

$$h_{\bigcap_{i=1}^n V_i} \in \bigcap_{i=1}^n C(V_i).$$

Thus the sets $C(V)$ satisfy the finite intersection property, so, because the sets $C(V)$ are closed and X is compact, Theorem 19 says that we can find some

$$h \in \bigcap_{V \in \mathcal{U}} C(V).$$

We will use h to construct an outer measure, which, restricted to Borel sets, will be our desired Haar measure. First, we need to prove a few useful properties of h.

Lemma 48. *Let $K, K_1, K_2 \subset G$ be compact, and let $g \in G$. Then,*

(1) $h(gK) = h(K)$.
(2) If $K_1 \subset K_2$, $h(K_1) \subset h(K_2)$.
(3) $h(K_1 \cup K_2) \leq h(K_1) + h(K_2)$.
(4) If $K_1 \cap K_2 = \varnothing$, then $h(K_1 \cup K_2) = h(K_1) + h(K_2)$.
(5) $h(K_0) = 1$.

Proof.

(1) Define a function $f : X \to \mathbb{R}$ by $f = \pi_{gK} - \pi_K$, where $\pi_K : X \to [0, (K : K_0)]$ denotes the projection map. Then, f is continuous by Theorem 7 because it is the composition of $g : X \to \mathbb{R}^2$ defined by $g(x) = (\pi_{gK}(x), \pi_K(x))$ (which is continuous by Theorem 22) and the continuous function $\mathbb{R}^2 \to \mathbb{R}$ sending $(x, y) \to x - y$. Further, because $h_U(gK) = h_U(K)$ for all $U \in \mathcal{U}$ by Lemma 47, we have $f(x) = 0$ for all $x \in \{h_U \mid U \in \mathcal{U}\}$. In particular, for each $V \in \mathcal{U}$, f is zero on the set

$$\{h_U \mid U \in \mathcal{U}, U \subset V\}.$$

Since $C(V)$ is the closure of this set, continuity of f implies that f is zero on all of the sets $C(V)$. Finally, since h is in the intersection of the sets $C(V)$, it follows that $h(gK) - h(K) = f(h) = 0$, giving $h(gK) = h(K)$.

(2) Define a function $f : X \to \mathbb{R}$ by $f = \pi_{K_2} - \pi_{K_1}$. Then, f is continuous by an argument similar to the one in part (1). Further, because $h_U(K_1) \le h_U(K_2)$ for all $U \in \mathcal{U}$, f is nonnegative on each set

$$\{h_U \mid U \in \mathcal{U}, U \subset V\}.$$

Since $C(V)$ is the closure of this set, continuity of f implies that f is nonnegative on all of the sets $C(V)$. Finally, since h is in the intersection of the sets $C(V)$, it follows that $h(K_1) - h(K_2) = f(h) \ge 0$, giving $h(K_1) \le h(K_2)$.

(3) Define a function $f : X \to \mathbb{R}$ by $f = \pi_{K_1} + \pi_{K_2} - \pi_{K_1 \cup K_2}$. It follows that f is continuous by an argument analogous to that of the previous parts. Further, by Lemma 47, for all $V \in \mathcal{U}$, f is nonnegative on the set $\{h_U \mid U \in \mathcal{U}, U \subset V\}$, and hence f is nonnegative on $C(V)$ by continuity. Thus, $f(h) \ge 0$ which is the same as saying $h(K_1 \cup K_2) \le h(K_1) + h(K_2)$.

(4) Suppose that $K_1 \cap K_2 = \varnothing$. By Theorem 16, we can find disjoint open sets $U_1 \supset K_1$ and $U_2 \supset K_2$. Now, by Theorem 37, we can choose $V_1 \in \mathcal{U}$ such that $K_1 V_1 \subset U_1$, and $V_2 \in \mathcal{U}$ such that $K_2 V_2 \subset U_2$. Let $V = V_1 \cap V_2$, which is an open set containing the identity, so $V \in \mathcal{U}$. Further, we have both $K_1 V \subset U_1$ and $K_2 V \subset U_2$, which means $K_1 V \cap K_2 V = \varnothing$. If $U \in \mathcal{U}$ is any subset of V^{-1}, we clearly have $K_1 U^{-1} \cap K_2 U^{-1} = \varnothing$, so $h_U(K_1 \cup K_2) = h_U(K_1) + h_U(K_2)$ by Lemma 47. Like before, we define a continuous function $f : X \to \mathbb{R}$ by $f = \pi_{K_1} + \pi_{K_2} - \pi_{K_1 \cup K_2}$. Noting that V^{-1} is open because the map $x \to x^{-1}$ is continuous (and thus a homeomorphism), we have shown that f is identically 0 on the set $\{h_U \mid U \in \mathcal{U}, U \subset V^{-1}\}$. Continuity of f implies that f is identically 0 on the closure of this set, which is $C(V^{-1})$. By definition, we have $h \in C(V^{-1})$, so $h(K_1 \cup K_2) = h(K_1) + h(K_2)$.

(5) Clearly $h_U(K_0) = 1$ for all $U \in \mathcal{U}$. Let $f = \pi_{K_0} : X \to [0, 1]$ denote the projection map. Then f is continuous and 1 on the set $\{h_U \mid U \in \mathcal{U}\}$. It follows that for all $V \in \mathcal{U}$, $f(x) = 1$ for all $x \in C(V)$. Thus $h(K_0) = f(h) = 1$.

□

We will now use h to define a function μ^* on all subsets of G, which will be an outer measure. First, for an open set $U \subset G$ we define

$$\mu^*(U) = \sup\{h(K) \mid K \subset U, K \text{ compact}\},$$

if U is nonempty and $\mu^*(\varnothing) = 0$. We note that for nonempty open sets U, the set $\{h(K) \mid K \subset U, K \text{ compact}\}$ is nonempty because singleton sets are always compact in a locally compact Hausdorff space.

Now, note that μ^* is monotonic on open sets, meaning if $U_1 \subset U_2$, then $\mu^*(U_1) \le \mu^*(U_2)$. Indeed, this follows from the fact that if $U_1 \subset U_2$ then $\{h(K) \mid K \subset U_1, K \text{ compact}\} \subset \{h(K) \mid K \subset U_2, K \text{ compact}\}$.

Now, if A is an arbitrary subset of G, we define

$$\mu^*(A) = \inf\{\mu^*(U) \mid A \subset U, U \text{ open}\}.$$

If U is open we have actually defined $\mu^*(U)$ in two different ways, however it is easy to see that these definitions agree using the fact that μ^* is monotonic on open sets. Also, note that $\mu^*(A_1) \leq \mu^*(A_2)$ if $A_1 \subset A_2$ trivially holds for all $A_1, A_2 \subset G$. We now show μ^* is an outer measure.

Lemma 49. *μ^* is an outer measure, meaning $\mu^* : \mathcal{P}(G) \to [0, \infty]$, where $\mathcal{P}(G) = \{X \mid X \subset G\}$, and*

(1) $\mu^*(\varnothing) = 0$,
(2) If $A_1 \subset A_2$ then $\mu(A_1) \leq \mu(A_2)$
(3) For any subsets $A_1, A_2, \ldots,$ of G,

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

Proof. We first need to check that μ^* is nonnegative. Clearly $h \geq 0$ because $h \in X$. It then follows trivially from the definition of μ^* that μ^* is nonnegative. Thus, μ^* is a function $\mathcal{P}(G) \to [0, \infty]$. (1) holds because $h \in X$ and by the definition of X this means $h(\varnothing) \in [0, (\varnothing : K_0)] = \{0\}$. We have already seen that (2) holds, so it only remains to show (3).

We first prove this for open sets. Let $U_1, U_2, \ldots,$ be a countable collection of open subsets of G. Let K be any compact subset of the open set $\bigcup_{i=1}^{\infty} U_i$. By compactness, we can find an integer $n > 0$ such that

$$K \subset \bigcup_{i=1}^{n} U_i.$$

Now, applying Lemma 26 inductively, we can find compact sets $K_i \subset U_i$ such that $K = \bigcup_{i=1}^{n} K_i$. Applying the definition of μ^* on open sets and part (3) of Lemma 48 inductively, we get

$$h(K) = h\left(\bigcup_{i=1}^{n} K_i\right) \leq \sum_{i=1}^{n} h(K_i) \leq \sum_{i=1}^{n} \mu^*(U_i) \leq \sum_{i=1}^{\infty} \mu^*(U_i).$$

Since this holds for all compact $K \subset \bigcup_{i=1}^{\infty} U_i$ and

$$\mu^*\left(\bigcup_{i=1}^{\infty} U_i\right) = \sup\{\mu(K) \mid K \subset \bigcup_{i=1}^{\infty} U_i, K \text{ compact}\}$$

by definition, we conclude

$$\mu^*\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \mu^*(U_i).$$

Now let $A_1, A_2, \ldots$ be any subsets of G and let $\epsilon > 0$ be given. By definition of μ^*, for each $i = 1, 2, \ldots$ we can find an open set $U_i \supset A_i$ such that $\mu^*(U_i) \leq \mu^*(A_i) + \frac{\epsilon}{2^i}$. Then, $\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} U_i$ so monotonicity of μ^* gives

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \mu^*\left(\bigcup_{i=1}^{\infty} U_i\right) \leq \sum_{i=1}^{\infty} \mu^*(U_i) \leq \sum_{i=1}^{\infty} \left(\mu^*(A_i) + \frac{\epsilon}{2^i}\right) = \left(\sum_{i=1}^{\infty} \mu^*(A_i)\right) + \epsilon.$$

Since this holds for all $\epsilon > 0$, we have

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i),$$

as desired. □

It now follows that μ^* restricts to a measure on the σ-algebra of μ^*-measurable sets, where a set B is defined to be μ^* measurable if and only if for all $A \subset G$,

$$\mu^*(A) = \mu^*(A \cap B) + \mu^*(A \cap B^c).$$

We now show that every open set U is μ^* measurable. By the previous lemma, we have $\mu^*(A) \leq \mu^*(A \cap U) + \mu^*(A \cap U^c)$ for all $A \subset G$, so we only need to show the reverse inequality. If $\mu^*(A) = \infty$, $A \cap U = \varnothing$, or $A \cap U^c = \varnothing$ then this is obvious, so suppose not. Then, by definition of μ^* we can find an open set $V \supset A$ such that $\mu^*(V) \leq \mu^*(A) + \epsilon/3$. Next, since $V \cap U$ is open, the definition of μ^* implies we can find a compact set K such that $h(K) \geq \mu^*(V \cap U) - \epsilon/3$. Then since $V \cap K^c$ is open we can find a compact set L such that $h(L) \geq \mu^*(V \cap K^c) - \epsilon/3$. Noting that $V \cap U^c \subset V \cap K^c$, we have

$$h(L) \geq \mu^*(V \cap K^c) - \epsilon/3 \geq \mu^*(V \cap U^c) - \epsilon/3.$$

Since K and L are disjoint, $A \subset V$, and $K \cup L$ is a compact subset of V, we have

$$\begin{aligned}
\mu^*(A) &\geq \mu^*(V) - \epsilon/3 \\
&\geq h(K \cup L) - \epsilon/3 \\
&= h(K) + h(L) - \epsilon/3 \\
&\geq (\mu^*(V \cap U) - \epsilon/3) + (\mu^*(V \cap U^c) - \epsilon/3) - \epsilon/3 \\
&= \mu^*(V \cap U) + \mu^*(V \cap U^c) - \epsilon \\
&\geq \mu^*(A \cap U) + \mu^*(A \cap U^c) - \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this implies

$$\mu^*(A) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c),$$

so U is μ^* measurable. Since the μ^* measurable sets form a σ-algebra and the Borel σ-algebra is defined to be the smallest σ-algebra containing all open sets, it follows that every Borel set is μ^* measurable.

Finally, we let μ denote the restriction of μ^* to the Borel sets. We claim that μ is our desired Haar measure. Since every Borel set is μ^* measurable, μ is a measure. By definition we have

$$\mu(E) = \inf\{\mu(U) \mid E \subset U, U \text{ open}\}$$

for all Borel sets E, so μ is outer regular.

Next we will prove inner regularity. First, we note that if K is compact then for all open sets $U \supset K$, $h(K) \leq \mu(U)$, by definition of μ. Thus,

$$\mu(K) = \inf\{\mu(U) \mid U \supset K, U \text{ open}\} \geq h(K).$$

Now, if U is any open set it follows that

$$\mu(U) = \sup\{h(K) \mid K \subset U, \text{ compact}\} \leq \sup\{\mu(K) \mid K \subset U, \text{ compact}\}.$$

However, monotonicity of μ implies $\sup\{\mu(K) \mid K \subset U, \text{ compact}\} \leq \mu(U)$, so

$$\mu(U) = \sup\{\mu(K) \mid K \subset U, K \text{ compact}\},$$

so μ is inner regular.

Also, the condition $\mu(K) \geq h(K)$ for compact K implies μ is not identically 0 because

$$\mu(K_0) \geq h(K_0) = 1.$$

We now need to show μ is finite on compact sets, so let $K \subset G$ be any compact set. Since G is locally compact and Hausdorff, for each $x \in K$ we can choose some open neighborhood V_x of x such that $\overline{V_x}$ is compact. The collection $\{V_x\}_{x \in K}$ is then an open cover of K, so by compactness there exist finitely many points $x_1, x_2, \ldots, x_n \in K$ which satisfy $K \subset \bigcup_{i=1}^n V_{x_i}$. Let $V = \bigcup_{i=1}^n V_{x_i}$. Then, V is an open set and $\overline{V}$ is compact because $\overline{V} = \bigcup_{i=1}^n \overline{V_{x_i}}$ is the finite union of compact sets. Now, for any compact subset L of V, we have $L \subset \overline{V}$, so $h(L) \leq h(\overline{V})$ by the monotonicity of h (part (2) of Lemma 48). Thus, $h(\overline{V})$ is an upper bound on the set $\{\mu(L) \mid L \subset V, L \text{ compact}\}$, so

$$h(\overline{V}) \geq \sup\{h(L) \mid L \subset V, L \text{ compact}\} = \mu(V).$$

Finally, monotonicity of μ gives

$$\mu(K) \leq \mu(V) \leq h(\overline{V}).$$

This implies $\mu(K)$ is finite because h is always finite (since $h \in X$).

Lastly, since h is left-translation-invariant, we have

$$\begin{aligned}\mu(U) &= \sup\{h(K) \mid K \subset U, K \text{ compact}\} \\ &= \sup\{h(gK) \mid K \subset U, K \text{ compact}\} \\ &= \sup\{h(K) \mid K \subset gU, K \text{ compact}\} \\ &= \mu(gU),\end{aligned}$$

for all open sets U and $g \in G$. Thus, for all Borel sets E,

$$\begin{aligned}\mu(E) &= \inf\{\mu(U) \mid E \subset U, U \text{ open}\} \\ &= \inf\{\mu(gU) \mid E \subset U, U \text{ open}\} \\ &= \inf\{\mu(U) \mid gE \subset U, U \text{ open}\} \\ &= \mu(gE).\end{aligned}$$

This shows that μ is left-translation-invariant, so it follows that μ is a left Haar measure. □

We will now show that the Haar measure on a locally compact group is essentially unique. To prove this, we will show that if μ and ν are Haar measures, then the corresponding integrals $\int_G f\,d\mu$ and $\int_G f\,d\nu$ are equal up to a constant, for $f \in C_c(G)$. We will then apply the Riesz Representation Theorem (Theorem 42). However, to work with these integrals we will need a version of Fubini's Theorem for measure spaces. To prove this theorem, the following lemma will be helpful.

Lemma 50. *Let S and T be topological spaces with T compact, and let $f : S \times T \to \mathbb{C}$ be a continuous function. Then, for each $\epsilon > 0$ and each $s_0 \in S$, there exists an open neighborhood U of s_0 such that $|f(s,t) - f(s_0,t)| < \epsilon$ for all $s \in U$ and $t \in T$.*

Proof. By continuity of f, for each $t \in T$ we can choose open neighborhoods U_t of s_0 and V_t of t such that for all $(s,t') \in U_t \times V_t$, $|f(s,t') - f(s_0,t)| < \epsilon/2$. It then follows that for all $(s,t') \in U_t \times V_t$, we have

$$|f(s,t') - f(s_0,t')| \leq |f(s,t') - f(s_0,t)| + |f(s_0,t) - f(s_0,t')| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Now, the sets V_t for $t \in T$ form an open cover of T, so by compactness of t, we can find finitely many points $t_1, \ldots, t_n \in T$ such that $T \subset \bigcup_{i=1}^n V_{t_i}$. We then define $U = \bigcap_{i=1}^n U_{t_i}$, which we claim satisfies the desired property. Indeed, U is an open neighborhood of s_0, and for all $(s,t) \in U \times T$, $t \in V_{t_i}$ for some i. Thus, $(s,t) \in U_{t_i} \times V_{t_i}$ so applying the inequality above gives

$$|f(s,t) - f(s_0,t)| < \epsilon.$$

□

For convenience in the following theorem, we make the following definition.

Definition 51. Let X and Y be sets and let f be a function on $X \times Y$. For each $x \in X$ and $y \in Y$, the *sections* f_x and f^y are defined to be the functions on Y and X, respectively, given by $f_x(y) = f(x,y)$ and $f^y(x) = f(x,y)$.

Theorem 52. *Let X and Y be locally compact Hausdorff topological spaces, let μ and ν be regular Borel measures on X and Y, respectively, and let $f \in C_c(X \times Y)$.*

(1) *For each $x \in X$ and $y \in Y$, $f_x \in C_c(Y)$ and $f^y \in C_c(X)$.*
(2) *Let $\phi_X : X \to \mathbb{C}$ be the function*

$$\phi_X(x) = \int_Y f_x(y)\, d\nu(y)$$

and let $\phi_Y : Y \to \mathbb{C}$ be defined by

$$\phi_Y(y) = \int_X f^y(x)\, d\mu(x).$$

Then $\phi_X \in C_c(X)$ and $\phi_Y \in C_c(Y)$.
(3) *The equality*

$$\int_X \int_Y f(x,y)\, d\nu(y) d\mu(x) = \int_Y \int_X f(x,y)\, d\mu(x) d\nu(y)$$

holds.

Proof.

(1) Let $K \subset \mathbb{C}$ denote the support of f, and let K_X and K_Y denote the image of K under the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, respectively. Then K_X and K_Y are compact since they are the continuous image of the compact set K. To prove the section f_x is continuous, we note that it is the composition of the map $y \to (x,y)$, which is continuous by Theorem 22, and f, which is continuous by assumption. Next, we note that if $f_x(y) = 0$ then $(x,y) \in K$, so $y \in K_Y$. Thus $\text{supp}(f_x) \subset K_Y$. Since $\text{supp}(f_x)$ is closed by definition, it is compact as a closed subset of a compact set. It follows that $f_x \in C_c(Y)$. Similarly, we have $f^y \in C_c(X)$.
(2) By part (1), $f_x \in C_c(Y)$ and $f^y \in C_c(X)$ so f_x is ν-integrable and f^y is μ-integrable and thus the functions ϕ_X and ϕ_Y are well defined everywhere. To prove continuity of ϕ_X, let $x_0 \in X$ and $\epsilon > 0$ be arbitrary. Then, by Lemma 50 applied to the function f and the set $X \times K_Y$, we can find an open neighborhood U of x_0 such that $|f_x(y) - f_{x_0}(y)| = |f(x,y) - f(x_0,y)| < \epsilon$

for all $x \in U$ and $y \in K_Y$. Then,

$$\begin{aligned}|\phi_X(x) - \phi_X(x_0)| &= \left|\int_Y f_x(y)\,d\nu(y) - \int_Y f_{x_0}(y)\,d\nu(y)\right| \\ &= \left|\int_{K_Y} (f_x(y) - f_{X_0}(y))\,d\nu(y)\right| \\ &\leq \int_{K_Y} |f_x(y) - f_{x_0}(y)|\,d\nu(y) \\ &\leq \epsilon\nu(K_Y),\end{aligned}$$

where we used the fact that for all y outside of K_y, $f_x(y) = f_{x_0}(y) = 0$. Since $\epsilon > 0$ was arbitrary, this implies ϕ_X is continuous at x_0. Since x_0 was arbitrary, ϕ_X is continuous on X. Also, we note that if $x \notin K_X$ then $f_x(y) = 0$ for all y and thus $\phi_X(x) = \int_Y 0\,d\nu = 0$. It follows that $\text{supp}(\phi_X) \subset K_X$ and thus $\text{supp}(\phi_X)$ is compact as a closed subset of a compact set. Putting everything together gives $\phi_X \in C_c(X)$. By the exact same argument, $\phi_Y \in C_c(Y)$.

(3) Part (2) shows that ϕ_X and ϕ_Y are μ-integrable and ν-integrable, respectively. We need to show that their integrals are equal. Let $\epsilon > 0$ be given. For each $x \in X$, we apply Lemma 50 to get an open neighborhood U_x of x such that $|f(x', y) - f(x, y)| < \epsilon$ for all $x' \in U$ and $y \in K_Y$. By compactness of K_X, we can find finitely many points $x_1, \ldots, x_n$ such that the sets $U_{x_1}, \ldots, U_{x_n}$ cover K_X. We now define sets A_i for $i = 1, \ldots, n$ by $A_1 = K_X \cap U_{x_1}$ and

$$A_i = K_X \cap U_{x_i} \cap (U_{x_{i-1}})^c \cap \cdots \cap (U_{x_1})^c$$

for $i = 2, \ldots, n$. The sets A_i are then disjoint Borel sets such that $A_i \subset U_{x_i}$ for all i and $K_X = \bigcup_{i=1}^n A_i$. Now define a function $g : X \times Y \to \mathbb{C}$ by

$$g(x, y) = \sum_{i=1}^n \chi_{A_i}(x) f(x_i, y).$$

We note that

$$\begin{aligned}\int_X \int_Y g(x, y)\,d\nu(y) d\mu(x) &= \int_X \sum_{i=1}^n \chi_{A_i}(x) \left(\int_Y f(x_i, y) d\nu(y)\right) d\mu(x) \\ &= \left(\int_Y f(x_i, y)\,d\nu(y)\right) \left(\int_X \sum_{i=1}^n \chi_{A_i}(x) d\mu(x)\right) \\ &= \int_Y f(x_i, y) \left(\int_X \sum_{i=1}^n \chi_{A_i}(x) d\mu(x)\right) d\nu(y) \\ &= \int_Y \int_X g(x, y)\,d\mu(x) d\nu(y).\end{aligned}$$

Next, note that f and g vanish outside of $K_X \times K_Y$. Also, for all $(x, y) \in K_X \times K_Y$ we have $x \in A_i$ for some i and thus and $|f(x, y) - g(x, y)| =$

$|f(x,y) - f(x_i,y)| < \epsilon$ since $x \in U_{x_i}$. Thus, we have

$$\begin{aligned}\left|\int_X \int_Y f(x,y)\,d\nu(y)d\mu(x) - \int_X \int_Y g(x,y)\,d\nu(y)d\mu(x)\right| \\ &= \left|\int_X \int_Y (f(x,y) - g(x,y))\,d\nu(y)d\mu(x)\right| \\ &= \left|\int_{K_X} \int_{K_Y} (f(x,y) - g(x,y))\,d\nu(y)d\mu(x)\right| \\ &\leq \int_{K_X} \int_{K_Y} |f(x,y) - g(x,y)|\,d\nu(y)d\mu(x) \\ &\leq \int_{K_X} \int_{K_Y} \epsilon\,d\nu(y)d\mu(x) \\ &= \epsilon\mu(K_X)\nu(K_Y)\end{aligned}$$

and similarly

$$\left|\int_Y \int_X f(x,y)\,d\mu(x)d\nu(y) - \int_Y \int_X g(x,y)\,d\mu(x)d\nu(y)\right| \leq \epsilon\mu(K_X)\nu(K_Y).$$

However, we showed that these two integrals of g are equal to each other, so the triangle inequality gives

$$\left|\int_X \int_Y f(x,y)\,d\nu(y)d\mu(x) - \int_Y \int_X f(x,y)\,d\mu(x)d\nu(y)\right| \leq 2\epsilon\mu(K_X)\nu(K_Y).$$

Since ϵ was arbitrary, this gives

$$\int_X \int_Y f(x,y)\,d\nu(y)d\mu(x) = \int_Y \int_X f(x,y)\,d\mu(x)d\nu(y),$$

as desired.

□

We are finally ready to prove the uniqueness of the Haar measure.

Theorem 53. *Let μ and ν be two Haar measures on a locally compact group G. Then there exists a positive real number c such that $\mu(E) = c\nu(E)$ for all Borel sets E.*

Proof. First, since μ is nonzero, we can find some set of positive measure. By outer regularity, it follows that we can find an open set with positive measure. Then by inner regularity, we can find a compact set K of positive measure. Also, $\mu(K)$ is finite by the regularity of μ.

Now let $g \in C_c(G)$ be a nonnegative real function, not identically 0. We show that $\int_G g\,d\mu > 0$. Let $U = g^{-1}(\mathbb{R}_{>0})$, which is open by continuity of g. Next, the collection of translates $\{gU\}_{g\in G}$ forms an open cover of K, so we can find some $g_1, \ldots, g_n$ such that $K \subset \bigcup_{i=1}^n g_iU$. We thus have

$$\mu(K) \leq \mu\left(\bigcup_{i=1}^n g_iU\right) \leq \sum_{i=1}^n \mu(gU) = n\mu(U)$$

by translation invariance of μ. Thus, $\mu(U) \geq \mu(K)/n > 0$. Noting that $U = \bigcup_{i=1}^\infty \{x \in G \mid g(x) > \frac{1}{i}\}$, it follows that we can find some i such that the set

$V = \{x \in G \mid g(x) > \frac{1}{i}\}$ has positive measure. Then, we have

$$\int_G g\,d\mu \geq \int_V g\,d\mu \geq \int_V \frac{1}{i}d\mu = \frac{\mu(V)}{i} > 0,$$

as desired.

Now, let $f \in C_c(G)$ be arbitrary. We will show that the ratio $\int_G f\,d\mu / \int_G g\,d\mu$ does not depend on the Haar measure μ, so that we have $\int_G f\,d\mu / \int_G g\,d\mu = \int_G f\,d\nu / \int_G g\,d\nu$. We start by defining a function $h : G \times G \to \mathbb{C}$ by

$$h(x, y) = \frac{f(x)g(yx)}{\int_G g(tx)\,d\lambda(t)},$$

where λ is any left Haar measure on G, possibly equal to μ or ν. The denominator of this is nonzero since, for all $x \in G$, $g(tx) \in C_c(G)$ and $g(tx)$ is nonnegative and nonzero. We now show that $h \in C_c(G \times G)$. Let $K = \text{supp}(f)$ and $L = \text{supp}(g)$. $h(x, y) \neq 0$ implies $f(x) \neq 0$ and $g(yx) \neq 0$, meaning $x \in K$ and $y \in Lx^{-1} \subset LK^{-1}$. We thus have $\text{supp}(h) \subset K \times LK^{-1}$. We show that LK^{-1} is compact as follows. K^{-1} is compact because it is the image of the compact set K under the continuous map $x \to x^{-1}$. Then, $L \times K^{-1}$ is compact by Theorem 24. The set LK^{-1} is then compact as it is the image of $L \times K^{-1}$ under the continuous map $(x, y) \to xy$. It follows by Theorem 24 that $K \times LK^{-1}$, and hence $\text{supp}(h)$, is compact.

To show that h is continuous, we first note that since g is continuous, $g(yx) : G \times G \to \mathbb{R}$ is continuous. Since f is continuous, the function $f(x)g(yx)$ is continuous as the product of two continuous complex-valued functions. We now show that the map $I(x) = \int_G g(tx)\,d\lambda(t)$ is continuous:

Lemma 54. *Let λ be a regular Borel measure on a locally compact group G. Then, the function $I : G \to \mathbb{C}$ given by $I(x) = \int_G g(tx)\,d\lambda(t)$ is continuous.*

Proof. As before we let L denote the support of g. Let $x_0 \in G$ be arbitrary and let $\epsilon > 0$ be given. By local compactness, we can find an open neighborhood W of x_0 with compact closure. It follows that the set $L\overline{W}^{-1}$ is compact in the same way we showed that LK^{-1} is compact, so $\lambda\left(L\overline{W}^{-1}\right)$ is finite. Thus we can choose some $\epsilon' > 0$ such that $\epsilon'\lambda\left(L\overline{W}^{-1}\right) < \epsilon$. Now, for each point $x \in L$, continuity of g implies we can choose some open neighborhood U_x of the identity such that

$$|g(x) - g(y)| < \frac{\epsilon'}{2}$$

for all $y \in xU_x$. Then, by Theorem 36, we can find an open set V_x containing the identity such that $V_xV_x \subset U_x$. By compactness of L, there are finitely many points $x_1, \ldots, x_n$ such that $L \subset \bigcup_{i=1}^n x_iV_{x_i}$. Define $V = \bigcap_{i=1}^n V_x$ and $U = V \cap V^{-1}$, which is an open neighborhood of the identity. We claim that for all $x, y \in G$ such that $y \in xU$,

$$|g(x) - g(y)| < \epsilon'.$$

Indeed, if x and y are both not in L, then clearly $|g(x) - g(y)| = 0 < \epsilon'$. Thus we may assume either $x \in L$ or $y \in L$. First consider the case $x \in L$. Then, $x \in x_iV_{x_i}$ for some i. Since $x_iV_{x_i} \subset x_iU_{x_i}$, $x \in x_iU_{x_i}$. Also, we have $y \in xU_x \subset xV_{x_i} \subset x_iV_{x_i}V_{x_i} \subset x_iU_{x_i}$. Thus, by definition of U_{x_i}, we have

$$|g(x) - g(y)| \leq |g(x) - g(x_i)| + |g(x_i) - g(y)| < \frac{\epsilon'}{2} + \frac{\epsilon'}{2} = \epsilon'.$$

Now consider the case $y \in L$. By assumption, $x \in yU$ so there exists some $u \in U$ such that $x = yu$. Rearranging, we get $y = xu^{-1}$. However, since $U = V \cap V^{-1}$, $u^{-1} \in U$ and thus $y \in xU$. It then follows by the previous case with x and y swapped that $|g(x) - g(y)| < \epsilon'$.

It now follows that for all $x \in W \cap x_0U$, we have $tx \in tx_0U$, implying $|g(tx) - g(tx_0)| < \epsilon'$. Thus, for all $x \in W \cap x_0U$,

$$\begin{aligned}
|I(x) - I(x_0)| &= \left| \int_G (g(tx) - g(tx_0))\, d\lambda(t) \right| \\
&\leq \int_G |g(tx) - g(tx_0)|\, d\lambda(t) \\
&= \int_{L\overline{W}^{-1}} |g(tx) - g(tx_0)|\, d\lambda(t) \\
&\leq \int_{L\overline{W}^{-1}} \epsilon'\, d\lambda(t) \\
&= \epsilon' \lambda \left(L\overline{W}^{-1} \right) \\
&< \epsilon,
\end{aligned}$$

where we used the fact that $g(tx) = g(tx_0) = 0$ if $t \notin L\overline{W}^{-1}$. Since $W \cap x_0U$ is an open neighborhood of x_0 and ϵ was arbitrary, it follows that I is continuous. □

Thus, h is continuous since it is a quotient of continuous complex-valued functions with nonzero denominator. We thus have $h \in C_c(G \times G)$, so we can apply Theorem 52. Because μ and λ are Haar measures, we can also use Theorem 38. Thus, we have

$$\begin{aligned}
\int_G \int_G h(x,y)\, d\lambda(y) d\mu(x) &= \int_G \int_G h(x,y)\, d\mu(x) d\lambda(y) \\
&= \int_G \int_G h(y^{-1}x, y)\, d\mu(x) d\lambda(y) \\
&= \int_G \int_G h(y^{-1}x, y)\, d\lambda(y) d\mu(x) \\
&= \int_G \int_G h(y^{-1}, xy)\, d\lambda(y) d\mu(x),
\end{aligned}$$

where we used Theorem 52 to swap the order of integration, then we used Theorem 38 to replace x with $y^{-1}x$, used Theorem 52 swap the order of integration again, and finally used used Theorem 38 again to replace y with xy. We now use the definition of h to compute these integrals. First, for all $x \in G$,

$$\int_G h(x,y)\, d\lambda(y) = \int_G \frac{f(x)g(yx)}{\int_G g(tx)\, d\lambda(t)}\, d\lambda(y) = f(x) \frac{\int_G g(yx)\, d\lambda(y)}{\int_G g(tx)\, d\lambda(t)} = f(x),$$

so

$$\int_G \int_G h(x,y)\, d\lambda(y) d\mu(x) = \int_G f\, d\mu.$$

From the definition of h, we get

$$h(y^{-1}, xy) = \frac{f(y^{-1})g(x)}{\int_G g(ty^{-1})\, d\lambda(t)}.$$

Then, for all $x \in G$,

$$\int_G h(y^{-1}, xy)\, d\lambda(y) = \int_G \frac{f(y^{-1})g(x)}{\int_G g(ty^{-1})\, d\lambda(t)}\, d\lambda(y) = g(x) \int_G \frac{f(y^{-1})}{\int_G g(ty^{-1})\, d\lambda(t)}\, d\lambda(y),$$

so

$$\int_G \int_G h(y^{-1}, xy)\, d\lambda(y) d\mu(x) = \int_G \frac{f(y^{-1})}{\int_G g(ty^{-1})\, d\lambda(t)}\, d\lambda(y) \int_G g\, d\mu.$$

Thus, we have

$$\begin{aligned}\int_G f\, d\mu &= \int_G \int_G h(x, y)\, d\lambda(y) d\mu(x) \\ &= \int_G \int_G h(y^{-1}, xy)\, d\lambda(y) d\mu(x) \\ &= \int_G \frac{f(y^{-1})}{\int_G g(ty^{-1})\, d\lambda(t)}\, d\lambda(y) \int_G g\, d\mu,\end{aligned}$$

which rearranges to

$$\frac{\int_G f\, d\mu}{\int_G g\, d\mu} = \int_G \frac{f(y^{-1})}{\int_G g(ty^{-1})\, d\lambda(t)}\, d\lambda(y)$$

because $\int_G g\, d\mu > 0$. By symmetry, the same result must hold for the measure ν, so

$$\frac{\int_G f\, d\nu}{\int_G g\, d\nu} = \int_G \frac{f(y^{-1})}{\int_G g(ty^{-1})\, d\lambda(t)}\, d\lambda(y),$$

giving

$$\frac{\int_G f\, d\mu}{\int_G g\, d\mu} = \frac{\int_G f\, d\nu}{\int_G g\, d\nu}.$$

Letting $c = \frac{\int_G g\, d\mu}{\int_G g\, d\nu}$, we get

$$\int_G f\, d\mu = c \int_G f\, d\nu$$

for all $f \in C_c(G)$. Now define a new measure ν' by $\nu'(E) = c\nu(E)$ for all Borel sets E. Clearly, we must have

$$\int_G f\, d\mu = \int_G f\, d\nu'$$

for all $f \in C_c(G)$. We can now define a positive linear functional Λ on $C_c(G)$ by

$$\Lambda(f) = \int_G f\, d\mu = \int_G f\, d\nu'.$$

However, by Theorem 42, there exists a unique regular Borel measure μ such that

$$\Lambda(f) = \int_G f\, d\mu$$

for all $f \in C_c(G)$. Since ν' is also a regular Borel measure satisfying this, it follows that we must have $\mu = \nu'$, so

$$\mu(E) = c\nu(E)$$

for all Borel sets E. □

This concludes the proof of the existence and uniqueness of a left Haar measure on any locally compact group G.

Remark 55. To prove uniqueness of the left Haar measure, we used the Riesz Representation Theorem (Theorem 42), whose proof was omitted because it is long and unrelated to the main content of this paper. However, we only needed to use the uniqueness part of this theorem, which can be proven easily using Theorem 43:

Theorem 56. *Let X be a locally compact Hausdorff topological space, and let Λ be a positive linear functional on $C_c(X)$. Then there exists at most one regular Borel measure μ such that*

$$\Lambda(f) = \int_X f \, d\mu$$

for all $f \in C_c(X)$.

Proof. Suppose μ and ν are two regular Borel measures both satisfying this. Let K be a compact set, which has finite μ-measure and ν-measure by regularity. Let $\epsilon > 0$ be given. Then, by outer regularity of ν, we can find some open set $V \supset K$ such that $\nu(V) \leq \nu(K) + \epsilon$. By Theorem 43, we can find some $f \in C_c(X)$ such that $\chi_K(x) \leq f(x) \leq \chi_V(x)$ for all $x \in X$. We then have

$$\mu(K) = \int_X \chi_K \, d\mu \leq \int_X f \, d\mu = \Lambda(f) = \int_X f \, d\nu \leq \int_X \chi_V f \, d\nu = \nu(V) \leq \nu(K) + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this gives $\mu(K) \leq \nu(K)$. The reverse inequality follows by symmetry, so $\mu(K) = \nu(K)$ for all compact sets K. Regularity then implies that for any open set U,

$$\mu(U) = \sup\{\mu(K) \mid K \subset U, K \text{ compact}\} = \sup\{\nu(K) \mid K \subset U, K \text{ compact}\} = \nu(U)$$

and thus for any Borel set E,

$$\mu(E) = \inf\{\mu(E) \mid E \subset U, U \text{ open}\} = \inf\{\nu(E) \mid E \subset U, U \text{ open}\} = \nu(E).$$

□

6. Examples

In this last section, we give a couple simple examples and applications of the Haar measure.

Example 57. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the circle as a subset of the complex plane. Then S^1 is a group under multiplication and inherits a locally compact, Hausdorff topology as a subset of $\mathbb{C}$. Thus, there is a unique Haar measure μ on S^1. For each point $z = e^{i\theta} \in S^1$, translation by z is simply rotation by the angle θ. The condition of translation invariance,

$$\mu(e^{i\theta} E) = \mu(E),$$

then simply says that μ is invariant under rotations.

We remark how this relates to the motivation behind the Haar measure. As said in the introduction, the idea behind the Haar measure is that the group structure of a locally compact group gives a group of symmetries–the translations–and the Haar measure is the unique measure for which these symmetries are measure-preserving. Invariance under these symmetries can be interpreted as the Haar measure being uniform. In the case of S^1, the symmetries derived from the group structure are simply the rotations. A uniform measure on S^1 should certainly be invariant under these. What we have shown in Section 5 is that this condition is actually enough to define a unique measure on S^1.

To explicitly construct a Haar measure on S^1, we can define $f : [0, 2\pi) \to S^1$ by $f(\theta) = e^{i\theta}$ and then let

$$\mu(E) = \frac{1}{2\pi}\lambda(f^{-1}(E)),$$

where λ is the Lebesgue measure on $\mathbb{R}$. The factor $\frac{1}{2\pi}$ ensures that $\mu(S^1) = 1$, so that μ is a probability measure. In fact, on any compact group G, any Haar measure is finite and thus, by scaling, there exists a unique Haar measure on G which is also a probability measure.

Example 58. Let $\mathrm{GL}_n(\mathbb{R})$ denote the group of invertible $n \times n$ matrices of real numbers. Identifying $\mathrm{GL}_n(\mathbb{R})$ with a subset of $\mathbb{R}^{n^2}$, we can endow $\mathrm{GL}_n(\mathbb{R})$ with the subspace topology. $\mathrm{GL}_n(\mathbb{R})$ is then a locally compact group, so it has a unique Haar measure. One way to write this measure is

$$\mu(S) = \int_S \frac{1}{|\det(X)|^n}\, dX,$$

where dX denotes the Lebesgue measure on the subset of $\mathbb{R}^{n^2}$ identified with $\mathrm{GL}_n(\mathbb{R})$. The translation invariance of this measure follows from the change of variables formula for integration. For each $M \in \mathrm{GL}_n(\mathbb{R})$, we let $T_M : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ denote the translation map. Considering T_M as a linear map on $\mathbb{R}^{n^2}$, we see that $\det(T_M) = \det(M)^n$. Thus, by the change of variables formula,

$$\begin{aligned}
\mu(MS) &= \int_{MS} \frac{1}{|\det(X)|^n}\, dX \\
&= \int_S \frac{1}{|\det(MX)|^n} \cdot |\det(T)|\, dX \\
&= \int_S \frac{1}{|\det(M)|^n |\det(X)|^n} \cdot |\det(M)|^n\, dX \\
&= \int_S \frac{1}{|\det(X)|^n}\, dX \\
&= \mu(S).
\end{aligned}$$

We conclude with an application of the Haar measure to give a simple proof that there is no countably infinite compact (Hausdorff) group:

Theorem 59. *There are no countably infinite compact groups.*

Proof. For the sake of contradiction, suppose that G is a countably infinite compact group. Since G is compact, it is locally compact so it has a Haar measure μ. Further, $\mu(G) < \infty$ because G is compact. Also, because G is Hausdorff, the singleton set $\{g\}$ is closed for all $g \in G$, and thus $\{g\}$ is Borel for all $g \in G$. Any two singleton sets $\{g_1\}$ and $\{g_2\}$ are translates of each other, namely by $g_2 g_1^{-1}$, so translation invariance of μ implies $\mu(\{g_1\}) = \mu(\{g_2\})$. However, we have $G = \bigcup_{g\in G}\{g\}$ where the union is countable, so

$$\mu(G) = \sum_{g\in G} \mu(\{g\}) = \sum_{g\in G} \mu(\{e\}),$$

where e is the identity of G. If $\mu(\{e\}) > 0$ then this implies $\mu(G) = \infty$ which is impossible, so we must have $\mu(\{e\}) = 0$ and thus μ is zero for all singleton sets. However, any set $E \subset G$ can be written as a countable disjoint union of singleton sets, since G is countable, so $\mu(E) = 0$ for all Borel sets E. This means that μ is identically 0, which is a contradiction. □

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