Mandelbrot Set

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Introduction

The Mandelbrot set is generated with iteration. The *seed* of the iteration will be denoted x_0. Then, the function $x^2 + c$ can be applied to get

$$x_1 = x_0^2 + c.$$

Continuing from this,

$$x_2 = x_1^2 + c$$
$$x_3 = x_2^2 + c$$
$$\vdots$$

The numbers $x_0, x_1, x_2, \ldots$ are often of interest.

More formally,

Definition 0.1 (Mandelbrot set)**.** The *Mandelbrot set* M is the set of all values c for which the sequence

$$0, P_c(0), P_c(P_c(0)), \ldots$$

where

$$P_c : z \to z^2 + c$$

does not go to infinity for all $z \in \mathbb{C}$.

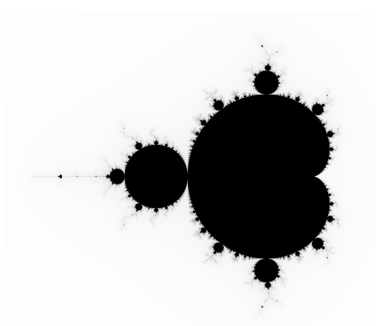

Figure 1: The Mandelbrot set.

Definition 0.2. Let $Q_c : z \mapsto z^2 + c$. Let K_c denote the filled Julia set of Q_c. The Mandelbrot set is the set of parameters c for which K_c is connected. This is equivalent to saying that $c \in M$ if and only if $0 \in K_c$. From this

$$Q_c^{\circ n}(0) \to \infty$$

if $c \notin M$.

Other terms relevant to the Mandelbrot set include:

Definition 0.3 (Bulb)**.** The main cardoid is the subset of the complex plane such that $c = \frac{1-(\mu-1)^2}{4}$ for some $\mu \leq 1$. The *bulb* at $c = \frac{-3}{4}$ is a circle with radius $\frac{1}{4}$ around -1. The bulb is the subset of the complex plane such that P_c has a cycle of period 2.

Definition 0.4 (Orbit)**.** The orbit is the sequence $z_0, z_1, z_2, \ldots$ where $z_n + 1 = P_c(z_n)$. The critical orbit is the orbit with $z = 0$. The fate of the critical orbit (whether it escapes to infinity or not) determines whether c is in the Mandelbrot set.

More Definitions

Definition 0.5. A function $f : \mathbb{C} \to \mathbb{C}$ is *complex differentiable* at $z \in \mathbb{C}$ if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for $h \in \mathbb{C}$. If f is complex differentiable at every $z \in U \subset \mathbb{C}$, then f is holomorphic on U.

Definition 0.6. Let $\Lambda, V \in \mathbb{R}^2$ denote two measurable sets. Additionally, let V satisfy $0 < \lambda(V) < \infty$. Define the *density* of Λ in V be

$$d_V(\Lambda) = \frac{\lambda(\Lambda \cap V)}{\lambda(V)}.$$

Definition 0.7. Let U be a connected open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ and $f : U \to \mathbb{C}$ be a holomorphic function. The distortion of f on U is defined as

$$\operatorname{dist}_U(f) = \sup_{x,y \in U} \left| \log \frac{f'(y)}{f'(x)} \right|.$$

Definition 0.8. A number $c \in \mathbb{C}$ is called a *critically non-recurrent* parameter is the critical point 0 of the quadratic Q_c does not belong to the closure of its forward orbit $\{Q_c^{\circ n}(0)\}_{n \geq 1}$.

Definition 0.9. Let M be a manifold. If the inner product $\langle \cdot, \cdot \rangle$ is defined on a tangent space $T_x M$ of M at every x, then the collection of these inner products is called the Riemannian metric.

Definition 0.10. f is expanding on Λ if there exists a neighborhood V of Λ in Ω and a continuous function $u : V \to \mathbb{R}_+^*$ such that f is strongly dilating on Λ for the Riemannian metric defined by u.

Definition 0.11. A holomorphic map f is strongly dilating on Λ for the Riemannian metric defined by u if there exists a $\lambda > 1$ such that for every $x \in \Lambda$

$$||T_x f|| = \frac{u_1(f(x))}{u(x)} |f'(x)| \leq \lambda.$$

The filled-in Julia set of a polynomial $f : \mathbb{C} \to \mathbb{C}$ of degree $d > 1$ is the set of points K_f of points z for which

$$f^{\circ n}(z) \not\to \infty.$$

Theorem 1. Let $f : U \to \mathbb{C}$ be a holomorphic map and let $\Lambda \subset U$ be a compact set such that $f(\Lambda) \subset \Lambda$ and f is expanding on Λ. Then, Λ has Lebesgue measure 0.

Proposition 2. Let $f : U \to \mathbb{C}$ be a holomorphic map. Let $\Lambda \subset U$ be a compact set and let $V \subset U$ be an open set such that $f_{|V}$ is injective. If $\mathrm{dist}_V(f) \leq m$,

$$1 - d_{f(V)} f(\Lambda) \leq e^{2m}(1 - d_V(\Lambda)).$$

Proof. Let $h = \inf_V |f'|$. Then,

$$\lambda(f(V)) \geq h^2 \lambda(V)$$

and

$$\lambda(f(V) \backslash f(\Lambda)) \leq \Lambda(f(V \backslash \Lambda)).$$

Additionally,

$$\begin{aligned} \lambda(f(V \backslash \Lambda)) &\leq h^2 e^{2m} \lambda(V \backslash \Lambda) \\ &= h^2 e^{2m} (1 - d_V(\Lambda)) \lambda(V). \end{aligned}$$

From this,

$$\begin{aligned} 1 - d_{f(V)} f(\Lambda) &= \frac{\lambda(f(V) \backslash f(\Lambda))}{\lambda(f(V))} \\ &\leq e^{2m}(1 - d_V(\Lambda)). \end{aligned}$$

□

Let $f : \Omega \to \Omega_1$ be a holomorphic map with $\Omega, \Omega_1 \subset \mathbb{C}$ being open subsets equipped with Riemannian metrics defined by u and u_1. For $x \in \Omega$ the norm of $T_x f : T_x \Omega \to T_{f(x)} \Omega_1$ is

$$||T_x f|| = \frac{u_1(f(x))}{u(x)} |f'(x)|.$$

Theorem 3. Let U be an open subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be a holomorphic map. Additionally, let $\Lambda \subset U$ be a compact set such that $f(\Lambda) \subset \Lambda$ for which f is expanding on Λ. Then, for every $m > 0$ there exist $a > 0$ and $b > 0$ for which all $\epsilon \in [0, a]$, $x \in \Lambda$, there exists an $n \in \mathbb{N}$ such that

$$B(f^{\circ n}(x) \subset f^{\circ n}(B(x, \epsilon)) \subset B(f^{\circ n}(x), b).$$

Lemma 4. The set Λ has empty interior.

Proof. The proof is omitted. □

Proof. Assume $\lambda > 0$. Choose $m > 0$ and let (ρ_n) be a sequence of positive numbers such that

$$\lim_{n\to\infty} \rho_n = 0.$$

By proposition 5.2, for each ν there exits a point $x_\nu \in \Lambda$ such that

$$d_{B(x_\nu,\rho_\nu)}(\Lambda) \to 1.$$

By theorem 5.1, there are numbers a and b and for each ν, $n_\nu \in \mathbb{N}$ such that

$$B(y_\nu, a) \subset f^{\circ n_\nu} B(x_\nu, \rho_\nu)$$

and

$$f^{\circ n_\nu} B(x_\nu, \rho_\nu) \subset B(y_\nu, b)$$

where $y_\nu = f^{\circ n_\nu} x_\nu$. Additionally,

$$\text{dist}_{B(x_\nu,\rho_\nu)}(f^{\circ n_\nu}) \leq m.$$

From this,

$$1 - d_{B(y_\nu,a)}(\Lambda) \leq \frac{b^2}{a^2}(1 - d_{f^{\circ n_\nu} B(x_\nu,\rho_\nu)}(\Lambda))$$

and

$$\frac{b^2}{a^2}(1 - d_{f^{\circ n_\nu} B(x_\nu,\rho_\nu)}(\Lambda)) \leq \frac{b^2}{a^2} e^{2m}(1 - d_{B(x_\nu,\rho_\nu)}(\Lambda)) \to 0.$$

Assume that

$$\lim_{n\to\infty} y_\nu = y$$

and

$$|y - y_\nu| < \frac{a}{2}$$

for all ν. Then, $B(y, \frac{a}{2}) \subset B(y_\nu, a)$ and

$$1 - d_{B(y,a/2)}(\Lambda) \leq 4(1 - d_{B(y_\nu,a)}(\Lambda)) \to 0.$$

Given this doesn't depend on ν,

$$d_{B(y,a/2)}(\Lambda) = 1$$

and

$$\Lambda \supset B(y, a/2)$$

since Λ is compact. By the previous lemma, this is a contradiction. □

Theorem 5. If f is hyperbolic,

$$\lambda(J_f) = 0.$$

Proof. This follows from the above theorem. $\square$

It is not known if $J_f = \partial K_f \cap \mathbb{R}$ has

$$\lambda(J_f) = 0$$

for every polynomial f.

Additionally, the set of parameters $c \in \partial M \cap \mathbb{R}$ for which the quadratic Q_c is critically non-recurrent has Lebesgue measure 0.