L^p Spaces

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Contents

1 L^p spaces

Occurring at the intersection of a multitude of motivations in mathematics, the L^p spaces find themselves a central object of study in many disciplines. The L^p spaces are vector spaces of functions, comprised of those measurable functions on a measure space $(X, \mathcal{A}, \mu)$ whose p-th power is (Lebesgue) integrable. The general L^p spaces generalize the notions of L^1, comprised simply of the integrable functions, and L^2, which the reader familiar with Fourier analysis may recognize as the space of square-integrable functions. It happens that the notion of the L^p spaces carries several nice properties that make it a focal object of study, which we will attempt to outline in this paper.

Definition 1.1 (L^p space)**.** For $1 \leq p < \infty$ and a measure space $(X, \mathcal{A}, \mu)$, the space $L^p(X, \mathcal{A}, \mu)$ consists of the measurable functions f that satisfy

$$\int_X |f|^p \, \mathrm{d}\mu < \infty.$$

When the underlying measure space $(X, \mathcal{A}, \mu)$ is understood, we write simply L^p.

We will eventually end up proving the L^p spaces are Banach spaces, i.e. complete normed vector spaces (see Definition 2.6). Firstly, a function space of functions from a set to a vector space receives a natural vector space structure given by pointwise addition and scalar multiplication, so we have the latter free of charge.

Thus it stands to define a norm on L^p, the notion of which we define more explicitly in Definition 2.1. For now, consider it a measure of the "magnitude" of the function in some fashion.

Definition 1.2 (L^p-norm)**.** The L^p-*norm* $\|f\|_p$ on measurable functions f is given by

$$\|f\|_p = \left[\int_X |f|^p \, \mathrm{d}\mu\right]^{\frac{1}{p}}.$$

The L^p norm is technically defined for $0 \leq p < 1$, but is not a valid norm as it fails the triangle inequality—see Definition 2.1 for an axiomatic definition of the norm. This is also why we restrict $p \geq 1$ in Definition 1.1. We will be working exclusively with $1 \leq p \leq \infty$ for the purposes of this paper.

We also allow functions to have $\|f\|_p = \infty$, so an equivalent definition for an L^p space is

$$L^p(X, \mathcal{A}, \mu) = \{f : X \to \mathbb{R} : f \text{ is measurable and } \|f\|_p < \infty\}.$$

Notice by definition, however, that $\|f\|_p = 0$ does not imply $f = 0$, but rather that $f = 0$ almost everywhere. This would fail one of the norm axioms, which dictates that the L^p norm must be zero if and only if $f = 0$. To alleviate this, the precise definition of L^p consists of *equivalence classes* of functions.

We simply define an equivalence relation on L^p in which $f \sim g$ when $f = g$ almost everywhere, because the Lebesgue integral ignores sets of measure zero. It follows that the only equivalence class of functions whose L^p norm is zero consists of exactly those functions that differ from $f = 0$ on a set of measure zero, so the original issue is no longer of concern. This consideration will rarely need to be accounted for, however, and we may work with elements of L^p as if they were functions.

1.1 Examples of finite L^p spaces

We provide some examples of common L^p spaces and their L^p-norms.

1.1.1 Common L^p norms

- $(\mathbb{R}^n, \mathcal{L}, \lambda)$. Perhaps the most common example of a measure space, let $\mathcal{L}$ be the Lebesgue σ-algebra and λ the Lebesgue measure. This produces a "continuous" L^p-space with the L^p-norm defined naturally as

$$\|f\|_p = \left[\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x\right]^{\frac{1}{p}}.$$

- $(\mathbb{Z}, \mathcal{M}, \mu)$. We may also take a "discrete" type of L^p space, defined on $X = \mathbb{Z}$ and with μ the counting measure. A measurable function on this measure space takes the form of a sequence of real numbers $\{a_n\}_{n\in\mathbb{Z}}$, where $f(x) = a_x$. Thus we may replace the integral in the L^p-norm with a sum, with the L^p-norm defined as

$$\|f\|_p = \left[\sum_{x=-\infty}^{\infty} |a_x|^p\right]^{\frac{1}{p}}.$$

1.1.2 Common values of p

- $L^1(X, \mathcal{A}, \mu)$. L^1 corresponds to the space of all integrable functions on $(X, \mathcal{A}, \mu)$; this is obviously useful for all sorts of applications that require integration.

- $L^2(X, \mathcal{A}, \mu)$. L^2 is notable for being the space of all *square-integrable* functions, which stems from issues in Fourier analysis. $p = 2$ is also a special case because it is its own Hölder conjugate: $\frac{1}{2} + \frac{1}{2} = 1$, and L^2 is unique for being the only L^p space that is also a Hilbert space (briefly, a complete vector space equipped with an inner product that induces the metric).

- $L^\infty(X, \mathcal{A}, \mu)$. L^∞ is the conjugate of L^1, see above. L^∞ also consists of the essentially bounded functions—those functions for which there exists a constant c such that the set of all x for which $f(x) > c$ has measure zero. See Subsection 1.2 for more details.

1.2 L^∞

We would like to be able to consider L^p spaces for all nonnegative p, but we run into some obstacles in doing so. We have already had to disregard L^p for $0 \leq p < 1$, as the L^p-norm fails an axiom; at risk of losing another useful notion, the case $p = \infty$ requires further care: how do we define the infinite power and $\frac{1}{\infty}$th power of something?

We have to construct a separate space for the case $p = \infty$ satisfying similar notions to those of L^p spaces for finite p. Recall that the Lebesgue integral of a measurable function f is the supremum of the integral of all simple functions strictly less than or equal to f; it follows that we would like to replace the notion of supremum with a construction better suited for the infinite case. In particular, we would like to ignore what the function does at a set of points of measure zero.

Definition 1.3 (Essential supremum)**.** The *essential supremum* is the smallest value that is greater than or equal to the function values everywhere except on a set of measure zero, i.e.

$$\text{ess sup}_X f = \inf\{a \in \mathbb{R} : \mu\{x \in X : f(x) > a\} = 0\}.$$

Definition 1.4 (L^∞ norm)**.** $\|f\|_\infty = \text{ess sup}_X |f|$.

We may then use this L^∞-norm in defining the space L^∞ in the same way we did in Section 1. As an example of why this definition is useful, see that the equivalence relation on L^p holds: the definition of essential supremum may be replaced by

$$\text{ess sup}_X f = \inf\{\sup_X g : g = f \text{ pointwise almost everywhere}\},$$

from which it follows that the equivalence relation on L^p, where we identify functions equivalent almost everywhere, is still valid.

2 Motivation and Applications

A driving question behind all mathematical studies is always that of importance—why should we care? It turns out that L^p spaces pop up when generalizing from several directions, and they are endowed with a variety of properties that we will find useful. This section will cover those motivating directions as well as the properties which make L^p spaces such focal objects of study.

2.1 From metric spaces

Viewing the L^p spaces as metric spaces under the metric induced by the L^p-norm positions them both as a natural generalization of the p-norm in finite Euclidean space and a special type of Banach space, both of which we will now explore.

2.1.1 The p-norm

In this section, we are assumed to be working in $\mathbb{R}^n$ unless stated otherwise. Consider the value of defining various notions of distance on a space, the motivation driving the study of metric spaces. For instance, take the Euclidean, taxicab, and Chebyshev metrics on $\mathbb{R}^2$, shown in Figure 1.

Each has different utilities—the Euclidean metric sees use most often in the plane with regards to geometry problems, the taxicab metric is so named for the number of blocks a taxi driver in a city with a grid layout would have to drive, and the Chebyshev metric can be thought of as the number of moves a king would have to make on a chessboard.

Consider their mathematical definitions in $\mathbb{R}^2$; suppose we are travelling a horizontal distance x and a vertical distance y. Then the distances are calculated as follows:

- Euclidean: $\sqrt{x^2 + y^2}$, "as the crow flies;"
- Taxicab: $x + y$, the sum of the horizontal and vertical distances;

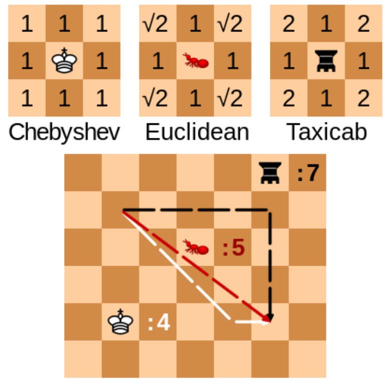

Figure 1. A comparison of the Chebyshev (king's move), Euclidean (shortest path), and taxicab (cardinal directions only) metrics.

- Chebyshev: $\max\{x, y\}$, whichever is longer.

We may desire to generalize these notions of distance. Rather than do this through a metric, we choose to do it through a *norm* that induces a metric; norms provide a generalized notion of magnitude rather than a specific notion of distance, the former of which will prove more useful for our purposes.

Definition 2.1. On a vector space V, a *norm* is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ satisfying the following properties:

1. Positive definiteness: $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$.
2. Absolute homogeneity: $\|s\vec{v}\| = |s| \cdot \|\vec{v}\|$ for all $\vec{v} \in V$ and scalars s.
3. Triangle inequality: $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ for all $\vec{v}, \vec{w} \in V$.

Definition 2.2. An *normed vector space* is a vector space equipped with a norm.

Notably, on a normed vector space, there is a naturally induced metric d defined by the norm of the difference of the two vectors: $d(\vec{v}, \vec{w}) = \|\vec{x} - \vec{y}\|$. Conversely, if a metric is absolutely homogeneous and translation invariant, meaning $d(\vec{v}, \vec{w}) = d(\vec{v} + \vec{a}, \vec{w} + \vec{a})$ for all vectors $\vec{a}$, then that metric is induced by the norm defined by $\|\vec{v}\| = d(\vec{v}, 0)$.

Note that the Euclidean, taxicab, and Chebyshev metrics satisfy the latter properties, and therefore have associated norms that induce them, named correspondingly. We may finally generalize the notion of distance, as desired, by generalizing the norms that induce the metrics we wanted. This is done through the *p-norm* $\|\vec{x}\|_p$, defined on a vector $\vec{x}$ as follows:

Definition 2.3 (The p-norm on vectors)**.** $\|\vec{x}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$.

For $0 \leq p < 1$, this does not define a norm, as it fails the triangle inequality; thus the p-norms are only valid norms for $p \geq 1$.

Notice that the Euclidean norm is the 2-norm, the taxicab norm is the 1-norm, and the Chebyshev norm is the limit of the p-norms as $p \to \infty$, often simply called the ∞-norm;

the corresponding metrics are induced similarly. Defining the general p-norm allows us to generalize this beyond these three special cases to any value of p we so desire.

This naturally extends to the space of measurable functions on $\mathbb{R}^n$, which we have already seen is endowed with a vector space structure. (There is an intermediate—the ℓ^p spaces, sequence spaces where the p-norm is extended to vectors of infinite length, but we will not cover those.) The analogy between the p-norm on vectors in $\mathbb{R}^n$ and the L^p norm on measurable functions can be seen as follows. First, rewrite the definition of the p-norm using summation notation:

$$\|\vec{x}\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}}.$$

We may visualize a function f as, in a sense, some kind of infinite-dimensional vector with entries corresponding to the values f takes on. Then we may replace the sum with the integral of f, such that

$$\|f\|_p = \left[\int_X |f|^p \, \mathrm{d}\mu\right]^{\frac{1}{p}}.$$

Of course, this notion is not entirely correct, but it serves to show how easily the L^p-norm, and thus the L^p spaces, arise from an elementary construction on metric spaces. Notice the other similarities as well—the triangle inequality failing for $0 \leq p < 1$, the significance of the cases $p = 1, 2, \infty$, and so on.

2.1.2 Banach spaces

As mentioned, L^p spaces are a class of *Banach spaces*, which are complete normed vector spaces. We have already introduced the concept of a normed vector space, so it stands to reintroduce completeness, for which Cauchy sequences are a prerequisite:

Definition 2.4 (Cauchy sequence)**.** In a metric space (X, d), a sequence $\{x_i\}_{i\in\mathbb{N}}$ is *Cauchy* if for every positive real $\varepsilon > 0$, there exists a positive integer N such that for all positive integers $m, n > N$, $d(x_m, x_n) < \varepsilon$.

Essentially, a Cauchy sequence is a sequence whose elements become arbitrarily close to each other as the indices tend to infinity. Completeness is then the requirement that every Cauchy sequence converges to an element in the metric space:

Definition 2.5 (Completeness)**.** A metric space (X, d) is *complete* if every Cauchy sequence in (X, d) converges to an element in X.

Definition 2.6 (Banach space)**.** A *Banach space* is a complete normed vector space.

We will show in Subsection 3.3 that L^p spaces are Banach spaces. A natural question is, of course, why does this matter? First of all, completeness is a very useful notion in and of itself—it allows us to define functions as limits of functions that we better understand, as opposed to having to work with some unwieldy and difficult-to-characterize function.

Banach spaces are also "complete" in another sense, in that they are equipped with most of the tools we would like to have in general: the norm allows us to compute vector magnitude (length), the induced metric allows us to compute distance, and completeness allows us to work more freely with limits and sequences. These characteristics make Banach spaces central objects of research across several areas of analysis, and that L^p spaces possess these characteristics make them amenable to more specific study as well. The fact that the L^p spaces are complete is actually a major component of why the Lebesgue integral is such a successful construction!

2.2 From other fields

While no other field has as direct an analog as the p- and ℓ^p-norm, other motivating factors pop up across much of mathematics. Several other fields whose study depends on that of L^p spaces, and thus provides a motivating factor for the study of L^p spaces, are listed below.

- Measure theory: another very natural way of reaching the L^p spaces, where—as mentioned several times—the L^p spaces are directly connected to issues with integrability.
- Fourier analysis: the Fourier transform is defined primarily on L^1 and L^2; specifically, the classical Fourier transform forms a map from $L^p \to L^p$ where $p \in [1, 2]$. Properties of L^1 and L^2 make results on Fourier transforms amenable, and in fact allow for some of the central results on the Fourier transforms of functions in $L^1 \cap L^2$ or L^2 alone.
- Nonlinear partial differential equations: much of the theory of these rests on the theory of L^p spaces, with results involving L^p spaces spanning topics from the 3D Navier-Stokes equations to images processing involving the p-Laplacian. Unfortunately, these results go far beyond the scope of what can be briefly summarized.
- Probability theory: the L^p norms yield the p-th moments of a random variable, which are quantitiative measures that contain information about the function's graph and distribution. Along with the 1st and 2nd moments, the 3rd and 4th moments (corresponding respectively to the L^3- and L^4-norms) appear nontrivially in many instances.

A common question is also the necessity of results on L^p for $p \neq 1, 2, \infty$, since those three appear so often and the others so rarely; it happens that many of the more advanced results, such as those in nonlinear partial differential equations, use other values of p liberally.

In addition, remaining within the context of L^p spaces, interpolation theorems such as those discussed in Subsection 4.2 provide an impetus for studying all exponents p, by allowing the transfer of some information from linear operators on functions in L^1 and L^∞ to operators on L^2 by way of all the intermediate exponents.

3 Fundamental Results on L^p Spaces

For all the setup and motivation, we still have yet to do anything with L^p spaces. Here we endeavor to prove the theorems that show L^p spaces behave the way we have said they do—in particular, we prove they are Banach spaces through a sequence of three core results.

3.1 Hölder's inequality

One of the most fundamental results on L^p spaces is Hölder's inequality, which may be thought of as an extension of the Cauchy-Schwarz inequality to the L^p spaces.

Theorem 3.1 (Hölder's inequality)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and take p, q where $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q, \tag{3.1}$$

with equality when $|f|^p$ and $|g|^q$ are linearly dependent in L^1.

The p, q above are called *Hölder conjugates*; we will use them again in some advanced results. We also have the convention that 1 is the conjugate of ∞ and vice versa. The inequality states that the norm of the (pointwise) product of two functions is bounded by the product of the norms of the two functions, as expected intuitively. To prove the result, we will need a lemma first, which in turn generalizes the well-known arithmetic-geometric mean (AM-GM) inequality:

Lemma 3.2. *Let $a, b \geq 0$ and $0 < \lambda < 1$. Then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda)b,$$

with equality only when $a = b$.

Proof. The result is obvious when $b = 0$, so we may disregard that case. When $b \neq 0$, we are able to divide both sides by b and introduce the change of variable $t = \frac{a}{b}$, such that we would like to show $t^\lambda \leq \lambda t + (1 - \lambda)$, with equality only when $t = 1$.

Rearranging the inequality gives us $t^\lambda - \lambda t \leq 1 - \lambda$; elementary techniques in calculus tell us that $t^\lambda - \lambda t$ is strictly increasing when $t < 1$ and strictly decreasing when $t > 1$, so its maximum value must occur at $t = 1$. This maximum value, plugging in $t = 1$, is exactly $1 - \lambda$—therefore we have equality in that case, and by how the function acts elsewhere, the inequality is strict for all other t. ∎

Proof of Hölder's inequality. We would like to reduce the cases we must consider. Firstly, note that we may disregard the cases for which $\|f\|_p = 0$ or $\|g\|_p = 0$ (since this implies they are zero almost everywhere), $\|f\|_p = \infty$, and $\|g\|_p = \infty$.

Also observe that the desired result is independent of scaling: if it holds for f, g, then it holds for $\alpha f, \beta g$ where α, β are scalars, as absolute homogeneity dictates we may take the scalars out of the norms and both sides are then scaled by the same amount. Thus we may prove only the case where $\|f\|_p = \|g\|_q = 1$, and show we have equality exactly when $|f|^p = |g|^q$ almost everywhere.

We apply Lemma 3.2 by choosing $a = |f|^p$, $b = |g|^q$, and $\lambda = \frac{1}{p}$ (note these are the absolute values, not the p/q-norms!), this particular choice of λ is chosen because then $1 - \lambda = \frac{1}{q}$ by definition. This yields the inequality

$$\begin{aligned}(|f(x)|^p)^{\frac{1}{p}}(|g(x)|^q)^{\frac{1}{q}} &\leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q \\ |f(x)g(x)| &\leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q.\end{aligned}$$

Integrating both sides, we see these absolute values come out to L^p-norms, which is useful for us because we set the L^p-norms of f and g to 1:

$$\int_X |f(x)g(x)|\,\mathrm{d}\mu \leq \frac{1}{p}\int_X |f|^p\,\mathrm{d}\mu + \frac{1}{q}\int_X |g|^q\,\mathrm{d}\mu$$
$$\|fg\|_1 \leq \frac{1}{p}\|f\|_p + \frac{1}{q}\|g\|_q$$
$$\|fg\|_1 \leq \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p\|g\|_q.$$

Finally, equality holds here if and only if it holds in the first inequality, which by Lemma 3.2 happens exactly when $|f|^p = |g|^q$ (almost everywhere). ■

The statement of "linear dependence" in the statement of Hölder's inequality really means that $\alpha|f|^p = \beta|g|^q$ almost everywhere for some scalars α, β, which should be easily seen from how we just proved equality holds and the earlier scaling argument.

Armed with Hölder's inequality, we may proceed to the first step in showing the L^p spaces are Banach spaces: proving they are normed vector spaces, i.e. that the L^p norm satisfies the norm axioms. Recall from Subsection 2.1.1 that these norm axioms are absolute homogeneity, positive definiteness, and the triangle inequality.

From the consideration we made in Section 1, where we defined L^p as the set of equivalence classes of p-integrable functions, as well as the fact that positivity is a given since we take the absolute value of the function inside the integral, we have positive definiteness.

Absolute homogeneity follows from the fact that we can take out a scalar s from the L^p-norm:

$$\|sf\|_p = \left[\int_X |sf|^p\right]^{\frac{1}{p}} = \left[\int_X |s|^p|f|^p\right]^{\frac{1}{p}} = \left[|s|^p\int_X |f|^p\right]^{\frac{1}{p}} = |s|\left[\int_X |f|^p\right]^{\frac{1}{p}}$$

and so it stands to prove the generalization of the triangle inequality on L^p, which is Minkowski's inequality.

3.2 Minkowski's inequality

Recall the triangle equality in Euclidean space, which geometrically states that the sum of the lengths of any two sides must be greater than or equal to the length of the remaining side, and which has the form of

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|.$$

With this in hand, Minkowski's inequality should look familiar:

Theorem 3.3 (Minkowski's inequality)**.** *Let $f, g \in L^p$ for $1 \leq p \leq \infty$. Then $f + g \in L^p$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \tag{3.2}$$

Proof. We trivially have the result for the cases $p = 1$ or $f + g = 0$ almost everywhere, so we may disregard those cases. When $p > 1$ and $f + g \neq 0$, we may write the inequality

$$\begin{aligned} |f + g|^p &\leq (|f| + |g|)|f + g|^{p-1} && (3.3) \\ &= |f||f + g|^{p-1} + |g||f + g|^{p-1}, && (3.4) \end{aligned}$$

since $|f| + |g| \geq |f + g|$. Notice also that $(p-1)q = p$ when q is conjugate to p, by definition, so that $|f + g|^{p-1} \in L^q$; then we may apply Hölder's inequality to the two terms on the right-hand side above to yield

$$|f| \in L^p,\ |f + g|^{p-1} \in L^q \implies \|\,|f| \cdot |f + g|^{p-1}\,\|_1 \leq \|f\|_p \cdot \|\,|f + g|^{p-1}\|_q$$

and likewise for g. Then in recognizing that the LHS of the above is the integral of what we have in (3.4), that

$$\|\,|f| \cdot |f + g|^{p-1}\,\|_1 = \int_X |f||f + g|^{p-1}\,\mathrm{d}\mu,$$

we may integrate (3.4) and make the substitution to yield

$$\begin{aligned} \int_X |f + g|^p\,\mathrm{d}\mu &\leq \|f\|_p \cdot \|\,|f + g|^{p-1}\|_q + \|g\|_p \cdot \|\,|f + g|^{p-1}\|_q \\ &= (\|f\|_p + \|g\|_p) \cdot \|\,|f + g|^{p-1}\|_q. \end{aligned}$$

Again, however, we invoke that $(p - 1)q = p$ on the far-right term, such that

$$\|\,|f + g|^{p-1}\|_q = \left[\int_X (|f + g|^{p-1})^q\right]^{\frac{1}{q}} \mathrm{d}\mu = \left[\int_X |f + g|^p\right]^{\frac{1}{q}} \mathrm{d}\mu.$$

Since this is guaranteed to be positive, we may divide both sides without flipping the sign:

$$\left[\int_X |f + g|^p\,\mathrm{d}\mu\right]^{1-\frac{1}{q}} \leq \|f\|_p + \|g\|_p.$$

But notice that $1 - \frac{1}{q} = \frac{1}{p}$, so the LHS is also $\|f + g\|_p$. Thus we have the desired result:

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$ ■

Therefore, by our arguments earlier about positive definiteness and absolute homogeneity, we get the corollary:

Corollary 3.4. *The L^p-norm is a norm on $L^p(X, \mathcal{A}, \mu)$.*

3.3 Riesz-Fischer's theorem

Minkowski's theorem just showed we can induce a metric on the L^p spaces by way of the L^p-norm, so we would like to finally be able to use this metric to show that L^p is complete.

To complete the proof that L^p spaces are Banach spaces after having shown that they are normed vector spaces, we must show they are complete—every Cauchy sequence of functions in L^p converges to a function in L^p. This result is commonly called Riesz-Fischer's theorem.

Theorem 3.5 (Riesz-Fischer)**.** *Every L^p space is complete under the L^p-norm.*

To prove this, we will need to recall the monotone and dominated convergence theorems from measure theory:

Theorem 3.6 (Monotone convergence)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f_1, f_2, \ldots$ a sequence of measurable functions from $X \to [0, \infty]$ such that for all $x \in X$, we have $0 \leq f_1(x) \leq f_2(x) \leq \cdots$. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Then*

$$\int_X f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu.$$

Theorem 3.7 (Dominated convergence)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space, define an integrable function $g : X \to [0, \infty]$ (i.e. $g \in L^1$), and let $f_1, f_2, \ldots$ be a sequence of measurable functions from $X \to [0, \infty]$ such that $lim_{n\to\infty} f_n(x)$ exists for each $x \in X$. Call this limit function f:*

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Suppose furthermore that $f_n(x) \leq g(x)$ for all n, x. Then f and all f_ns are integrable, such that

$$\int_X f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu.$$

We will also need a lemma on proving completeness. A series $\sum_1^\infty f_n$ *converges* to f if $\lim_{n\to\infty} \sum_1^n f_n = f$, and is *absolutely convergent* if $\sum_1^\infty \|f_n\| < \infty$.

Lemma 3.8. *A normed vector space V is complete if every absolutely convergent series in V converges.*

Proof. The converse is also true, but we won't need that result; we will also use notation as working in L^p in this proof. Let $\{f_n\}$ be a Cauchy sequence. Then in Definition 2.4, we may choose $\varepsilon = 2^{-j}$ and $N = n_j$, where we have an increasing sequence of integers $n_1 < n_2 < \cdots$, such that $\|f_n - f_m\|_p < 2^{-j}$ for all $m, n > n_j$, i.e. the bound between successive terms gets tighter as the indices tend to infinity.

Define another sequence $\{g_j\}$ such that $g_1 = f_{n_1}$ and $g_j = f_{n_j} - f_{n_j - 1}$ for $j > 1$. Then clearly the sum $\sum_1^k g_j = f_{n_k}$, and we have that

$$\sum_1^\infty \|g_j\|_p \leq \|g_1\|_p + \sum_1^\infty 2^{-j} = \|g\|_p + 1 < \infty,$$

the former following from our statement that $\|f_n - f_m\|_p < 2^{-j}$ for all $m, n > n_j$ and the latter following from the famous infinite sum $\sum_1^\infty 2^{-n} = 1$. Therefore $\lim_{n\to\infty} f_{n_k} = \sum_1^\infty g_j$ exists and is in V, and it is easy to see from the definition of a Cauchy sequence that its limit is the same as $\lim_{n\to\infty} f_n$. Therefore $\lim_{n\to\infty} f_n \in V$, and by Definition 2.5, we are done. ■

We are now equipped to prove Riesz-Fischer's theorem.

Proof of Riesz-Fischer. Let $\{f_k\} \in L^p$ be a Cauchy sequence, such that $\sum_1^\infty \|f_k\|_p = B < \infty$ for some B. Take the partial sums $G_n = \sum_1^n |f_k|$ and $G = \lim_{n\to\infty} G_n = \sum_1^\infty |f_k|$. It follows that $\|G_n\|_p \le \sum_1^n \|f_k\|_p \le B$ for all n, because

$$\left[\int_X ||f_1| + \cdots + |f_n||^p \, \mathrm{d}\mu\right]^{\frac{1}{p}} \le \left[\int_X |f_1|^p \, \mathrm{d}\mu\right]^{\frac{1}{p}} + \cdots + \left[\int_X |f_n|^p \, \mathrm{d}\mu\right]^{\frac{1}{p}} \tag{3.5}$$

and of course $\sum_1^n \|f_k\|_p \le \sum_1^\infty \|f_k\|_p = \infty$. Notice that the outer absolute value bars are irrelevant inside the integral in the LHS, so we may see that

$$\begin{aligned}\left[\int_X ||f_1| + \cdots + |f_n||^p \, \mathrm{d}\mu\right]^{\frac{1}{p}} &= \left[\int_X (|f_1| + \cdots + |f_n|)^p \, \mathrm{d}\mu\right]^{\frac{1}{p}} \le B \\ &= \left[\int_X G_n^p \, \mathrm{d}\mu\right]^{\frac{1}{p}} \le B \\ &\implies \int_X G_n^p \, \mathrm{d}\mu \le B^p.\end{aligned}$$

We may apply the Monotone Convergence Theorem with the sequence G_n and limiting function G to yield that

$$\int_X G^p \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X G_n^p \, \mathrm{d}\mu,$$

from which it follows clearly that G is p-integrable and thus $G \in L^p$. Between this and the fact that each $\int_X G_n^p \, \mathrm{d}\mu$ is bounded, we may easily see that $G(x) < \infty$ almost everywhere.

In particular, by our definition of G, this means that the series $\sum_1^\infty f_k$ converges almost everywhere (though note the distinction between $G = \sum_1^\infty |f_k|$ and the above). Let this sum be F, such that $F = \sum_1^\infty f_k$; we see immediately that $|F| \le G$ and therefore F is p-integrable, so $F \in L^p$.

We may also see that

$$\left|F - \sum_1^n f_k\right|^p \le (2G)^p \in L^1. \tag{3.6}$$

To see this, consider the alternative formulation

$$\left|\sum_n^\infty f_k\right|^p \le \left(2\sum_1^\infty |f_k|\right)^p \in L^1,$$

from which the inequality should be immediately evident. The fact that $(2G)^p \in L^1$ follows from the fact that $G \in L^p \implies G^p \in L^1$, i.e. if G is p-integrable then G^p is integrable.

Therefore our difference of series $|F - \sum_1^n f_k|^p \in L^1$ as well, so we may integrate it. Notice that this is closely related to the L^p-norm of the same function:

$$\int_X \left|F - \sum_1^n f_k\right|^p \mathrm{d}\mu = \left[\left[\int_X \left|F - \sum_1^n f_k\right|^p \mathrm{d}\mu\right]^{\frac{1}{p}}\right]^p = \left\|F - \sum_1^n f_k\right\|_p^p. \tag{3.7}$$

Applying the dominated convergence theorem to (3.6) with $g = G$ and $f_n = |F - \sum_1^n f_k|^p$ shows us:

$$\int_X \lim_{n\to\infty} \left| F - \sum_1^n f_k \right|^p \mathrm{d}\mu = \lim_{n\to\infty} \int_X \left| F - \sum_1^n f_k \right|^p \mathrm{d}\mu = \lim_{n\to\infty} \left\| F - \sum_1^n f_k \right\|_p^p,$$

but notice that the limit in the LHS tends to zero: $\lim_{n\to\infty} |F - \sum_1^n f_k|^p = |F - F|^p = 0$ by definition of F as the infinite sum of the f_k, so we have that

$$\lim_{n\to\infty} \left\| F - \sum_1^n f_k \right\|_p^p = 0 \implies \lim_{n\to\infty} \left\| F - \sum_1^n f_k \right\|_p = 0,$$

and therefore the series $\sum_1^\infty f_k$ is absolutely convergent in the L^p-norm. We showed earlier that $\sum_1^\infty f_k$ is convergent almost everywhere, so by Lemma 3.8, L^p is complete. ■

3.4 Density of simple functions

Having proven that L^p spaces are Banach spaces, we would like to prove other properties on them. We may prove results that hold for all $f \in L^p$ by proving them on a dense subspace of L^p and extending the result by continuity, rather than painstakingly proving them for the whole space. It happens that a very fortuitous subset of L^p is dense—the simple functions s, those that take on only a finite number of values, defined by the constant values they take on over each interval.

Theorem 3.9. *For $1 \le p \le \infty$, the simple functions in L^p are dense in L^p.*

Proof. We would like to show that we can approximate any function $f \in L^p$ by simple functions. Then there is a sequence $\{f_n\}$ of simple functions that converge to f pointwise almost everywhere and such that $|f_n| \le |f|$. Clearly each $f_n \in L^p$, and $|f - f_n|^p \le 2^p|f|^p \in L^1$ (with equality when $f_n = -f$), again since f being p-integrable implies $|f|^p$ is integrable.

Applying the dominated convergence theorem with $g = 2^p|f|^p$ and the $f_n = |f - f_n|^p$, justified as the f_n converge to f, tells us that

$$\begin{aligned} \int_X \lim_{n\to\infty} |f - f_n|^p \,\mathrm{d}\mu &= \lim_{n\to\infty} \int_X |f - f_n|^p \,\mathrm{d}\mu \\ 0 &= \lim_{n\to\infty} \|f - f_n\|_p^p, \end{aligned}$$

as the f_n converging to f pointwise means the internal limit on the LHS goes to infinity, and the RHS integral is the pth power of the L^p norm of $f - f_n$. Therefore any $f \in L^p$ can be approximated arbitrarily well by simple functions, so it follows that the simple functions are dense in L^p. ■

4 Advanced Results on L^p Spaces

Having established the fundamentals of L^p spaces, we now turn to addressing some of the more advanced results that make L^p unique and useful. We discussed some of their applications in Subsection 2.2, and these are some of the results that make L^p spaces so well suited to those applications.

4.1 Inclusions between L^p spaces

4.1.1 General inclusions

A natural question to ask is when a function $f \in L^p$ is also in L^q for some $q \neq p$. Naïvely one might expect this to have a simple answer, like expecting every square-integrable function in L^2 to also be integrable in L^1, but this happens to not be true, as seen by the following two examples:

Lemma 4.1. *Consider $f = \frac{1}{x}$ over $X = [1, \infty)$ equipped with the Lebesgue measure. Then $f \in L^2$ but $f \notin L^1$.*

Lemma 4.2. *Consider $f = \frac{1}{\sqrt{x}}$ over $X = [1, \infty)$ equipped with the Lebesuge measure. Then $f \in L^1$ but $f \notin L^2$.*

The integration of the function $\frac{1}{x}$ is a basic exercise in calculus classes, and its divergence in the integral to infinity is a product of its antiderivative being the natural logarithm; however, both $\frac{1}{x^2}$ and $\frac{1}{\sqrt{x}}$ converge as $x \to \infty$, going to zero. Since $L^1 \not\subset L^2$ and $L^2 \not\subset L^1$, there cannot be any general inclusion $L^p \subset L^q$.

However, we can find inclusion results between $L^p(X, \mathcal{A}, \mu)$ and $L^q(X, \mathcal{A}, \mu)$ given more restrictions on the underlying set X. For instance, the natural intuition is correct when X is finite:

Proposition 4.3. *If $\mu(X) < \infty$, then for $1 \leq p < q < \infty$, $L^q(X, \mathcal{A}, \mu) \subset L^p(X, \mathcal{A}, \mu)$.*

Proof. Take a function $f \in L^q$; we will show $f \in L^p$. The fact that the reverse inclusion does not hold should be obvious—consider extreme cases where $q \gg p$. We consider the functions $F = |f|^p$ and $G = 1$, with exponents $P = \frac{q}{p}$ (notice this is greater than 1) and Q its conjugate, i.e. $\frac{1}{P} + \frac{1}{Q} = 1$; we may then apply Hölder's inequality, which tells us that

$$\|FG\|_1 \leq \|F\|_P \|G\|_Q.$$

Taking $\|F\|_P$ is justified as $\|F\|_P = [|f|^p]^{\frac{q}{p}} = |f|^q$, which is valid as $f \in L^q$; obviously $\|G\|_Q$ is justified since $G = 1$ and 1 is everything-integrable. Expanding the above equation reveals that

$$\begin{aligned} \int_X FG \, \mathrm{d}\mu &\leq \left[\int_X |F|^P \, \mathrm{d}\mu \right]^{\frac{1}{P}} + \left[\int_X |G|^Q \, \mathrm{d}\mu \right]^{\frac{1}{Q}} \\ \int_X |f|^p \, \mathrm{d}\mu &\leq \left[\int_X |f|^{\frac{q}{p}} \, \mathrm{d}\mu \right]^{\frac{p}{q}} + \left[\int_X 1 \, \mathrm{d}\mu \right]^{1-\frac{p}{q}} \\ \|f\|_p^p &\leq \|f\|_q^q + \mu(X)^{1-\frac{p}{q}} < \infty. \end{aligned} \tag{4.1}$$

Quickly unpacking this, the second line is merely a restatement of the first. The changes in the third line follow from the fact that $\int_X |f|^{\frac{q}{p}} \, \mathrm{d}\mu \leq \int_X |f|^q \, \mathrm{d}\mu = \|f\|_q^q$ and that $\int_X 1 \, \mathrm{d}\mu = \mu(X)$ by definition. Since $f \in L^q$, $\|f\|_q < \infty$ and therefore its qth power is also finite. Since $\mu(X) < \infty$ by hypothesis, that term is also finite, so their sum is finite and therefore so is the L^p-norm of f; therefore, $f \in L^p$. ■

The above proof also nicely shows why this fails for $\mu(X) = \infty$; then the RHS of the final inequality would be infinity, so the L^p-norm of f could be infinite as well. There are also further results that we can generalize this to, given our intuition: consider $L^1(X, \mathcal{A}, \mu)$ and $L^\infty(X, \mathcal{A}, \mu)$ as comparative examples, where $X = [0, \infty)$ and $\mu = \lambda$. This provides us the useful framework of Riemann integration from single-variable calculus to work within, which will hopefully help the reader's intuition.

Here, functions in L^1 are permitted to blow up to infinity around 0, such as $\frac{1}{\sqrt{x}}$ as in the earlier example, but must decay as $x \to \infty$ to ensure the integral to infinity remains finite. On the contrary, functions in L^∞ are those for which there exists a finite value (the essential supremum) such that the function takes on higher values at a set of points that has measure zero; therefore such functions have no requirements on decay (i.e. constant functions $f(x) = c$, where obviously said finite value is just c), but cannot blow up to infinity anywhere, as otherwise the essential supremum would not exist.

Intuitively, as 1 and ∞ are the ends of the possible range of exponents p, one might expect some similar relation to hold for the intermediate exponents. And something similar indeed does hold:

Theorem 4.4. *Let $1 \leq p < q < \infty$. Then the following hold:*

1. *$L^q(X, \mathcal{A}, \mu) \subset L^p(X, \mathcal{A}, \mu)$ if and only if X does not contain sets of arbitrarily large finite measure.*

2. *$L^p(X, \mathcal{A}, \mu) \subset L^q(X, \mathcal{A}, \mu)$ if and only if X does not contain sets of arbitrarily small nonzero measure.*

Proof of (1). This is mostly an extension of Proposition 4.3. The if direction follows immediately from that proposition, considering that "contains sets of arbitrarily large finite measure" is equivalent to "has infinite measure" since there are no bounds on the measure of measurable sets.

The only if direction follows from some more intensive functional analysis. In particular, we require a result on Banach spaces called the *closed graph theorem*, which is stated as follows:

Theorem 4.5 (Closed graph theorem)**.** *A linear operator between Banach spaces is continuous if and only if the graph of that function is closed.*

In turn, the graph of a function f is the set $\{(x, f(x)) : x \in \text{dom } f\}$ where dom f is the domain of f, and closure here is in the sense of a topological space. We will use this result without proof. The idea is to show that the embedding operator that sends functions ins L^q to functions in L^p is continuous; this is one scenario in which completeness properties of Banach spaces become useful.

Let $\{f_n\}$ be a sequence of functions that converges to f in the L^q-norm, and to g in the L^p-norm. We can extract a subsequence of $\{f_n\}$ that converges almost everywhere to both f and g, which in turn implies $f = g$; this is done by taking a subsequence that converges almost everywhere to f, which will still converge to g in the L^p-norm, and then taking a subsequence of that subsequence that converges almost everywhere to g. This implies the set $\{(f, E(f)) : f \in L^q\}$ will be closed when E is the inclusion map embedding L^q into L^p, from which it follows that this embedding is continuous.

Continuity in this sense implies there exists some finite constant C such that $\|f\|_p \leq C\|f\|_q$. Since the underlying set X of the measure space $(X, \mathcal{A}, \mu)$ can be written as a limit of measurable sets of individually finite measure, suppose the sequence $\{A_n\} \subset \mathcal{A}$ where $\lim_{n\to\infty} A_n = X$, we may consider the indicator functions χ_{A_n} of these to yield that

$$\|\chi_{A_n}\|_p \leq C\|\chi_{A_n}\|_q \implies \mu(A_n)^{\frac{1}{p}} \leq C\mu(A_n)^{\frac{1}{q}}.$$

Isolating C yields that

$$\mu(A_n)^{\frac{q-p}{pq}} \leq C \implies \mu(A_n) \leq C^{\frac{pq}{q-p}},$$

since $p \neq q$. Taking the limit $n \to \infty$ yields that $\mu(X)$ is bounded by a finite value, and therefore $\mu(X) < \infty$, as desired. ∎

Proof of (2). First suppose $L^p \not\subset L^q$, such that there is $f \in L^p \setminus L^q$. Consider the set $A_M := \{x : |f(x)| \geq M\} \subset X$, i.e. the set of all points where f is greater than some fixed M. Since we have $\|f\|_p < \infty$ by construction, we may see that

$$\|f\|_p^p = \int_X |f|^p \, \mathrm{d}\mu \geq \int_{A_M} |f|^p \, \mathrm{d}\mu \geq M^p \mu(A_M).$$

Rearranging tells us that

$$\mu(A_M) \leq \|f\|_p^p \frac{1}{M^p},$$

which shows that $\mu(A_M) \to 0$ as $M \to \infty$. But if there is ever M such that $\mu(A_M) = 0$ exactly, then we must have that $f(x) < M$ for all x, which means by definition of the L^∞-norm that $\|f\|_\infty \leq M$. Then we can find a contradiction telling us $f \in L^q$ by way of proving its L^q-norm is finite: notice that

$$\|f\|_q^q = \int_X |f|^q \, \mathrm{d}\mu = \int_X |f|^p |f|^{q-p} \, \mathrm{d}\mu.$$

We can bound this using Hölder's inequality again, where $F = |f|^p$, $G = |f|^{q-p}$, and our exponents are the conjugates 1 and ∞; we then get that

$$\begin{aligned} \|f\|_q^q = \int_X |f|^p |f|^{q-p} \, \mathrm{d}\mu = \|FG\|_1 &\leq \|F\|_1 \|G\|_\infty \\ &\leq \int_X |f|^p \, \mathrm{d}\mu \cdot \operatorname{ess\,sup}_X |f|^{q-p} \\ &\leq \|f\|_p^p \|f\|_\infty^{q-p}, \end{aligned}$$

seeing as if M is the essential supremum of $|f|$, then we expect M^{q-p} to be the essential supremum of $|f|^{q-p}$, respectively. We know $\|f\|_p < \infty$ since $f \in L^p$, so we have that by definition; likewise, we just showed $\|f\|_\infty < \infty$ if we ever have $\mu(A_m) = 0$, so in that case we would have that $\|f\|_q \infty \implies f \in L^q$, a contradiction. So if X contains sets of arbitrary small nonzero measure, $L^p \not\subset L^q$; the contrapositive is what we desire, that $L^p \subset L^q$ if X does not contain sets of arbitrary small nonzero measure.

We also prove the other direction by contrapositive. Suppose X contains sets of arbitrarily small positive measure. Choose a measurable set B_1 such that $\mu(B_1) \in (0, 1]$ and inductively choose B_n such that $\mu(B_n) \in (0, \frac{1}{3}\mu(A_{n-1})]$. We may then find the following alternate expressions for the measure of B_n:

$$\mu(B_n) \in \left(0, \frac{1}{3^n}\right] \qquad \text{and} \qquad \mu(B_{n+k}) \in \left(0, \frac{1}{3^k}\mu(A_n)\right].$$

Now choose $C_n = B_n \setminus \bigcup_{k=1}^{\infty} B_{n+k}$, each of which has positive measure and is disjoint from all the other C_n. We can construct a simple function $f = \sum_n c_n \chi_{C_n}$ for a sequence c_n of values it takes on, where its L^p-norm can be given as

$$\|f\|_p^p = \int_X |f|^p \, \mathrm{d}\mu = \int_{C_1} |c_1|^p \, \mathrm{d}\mu + \int_{C_2} |c_2|^p \, \mathrm{d}\mu + \cdots = \sum_n c_n^p \mu(C_n).$$

Select $c_n = \mu(C_n)^{-\frac{1}{q}}$, so that

$$\|f\|_p^p = \int_X |f|^p \, \mathrm{d}\mu = \sum_n \mu(C_n)^{1-\frac{p}{q}} \leq \sum_n \frac{1}{3^{n\left(1-\frac{q}{p}\right)}}$$

since by construction $\mu(C_n) \leq \mu(B_n) \leq \frac{1}{3^n}$. Therefore our construction of $f \in L^p$. However, $f \notin L^q$:

$$\|f\|_q^q = \int_X |f|^q \, \mathrm{d}\mu = \sum_n \mu(C_n)^{1-\frac{q}{q}} = \sum_n 1 = \infty,$$

so the L^q-norm of f is infinite. Therefore if X contains sets of arbitrarily small positive measure, then $L^p \not\subset L^q$; the contrapositive tells us that if $L^p \subset L^q$, then X does not contain sets of arbitrarily small positive measure. ■

We have seen the former case with our proof that a finite measure space has $L^q \subset L^p$; an example in which $L^p \subset L^q$ is the integers $\mathbb{Z}$ equipped with the counting measure, in which case any p-integrable function must also be q-integrable.

4.1.2 Interpolative inclusions

We finish the section on L^p inclusions with some relations that will be instructive in the next section.

Theorem 4.6. *Let $1 \leq p < q < r \leq \infty$. Then the following hold:*

1. $L^p(X, \mathcal{A}, \mu) \cap L^r(X, \mathcal{A}, \mu) \subset L^q(X, \mathcal{A}, \mu)$.
2. $L^q(X, \mathcal{A}, \mu) \subset L^p(X, \mathcal{A}, \mu) + L^r(X, \mathcal{A}, \mu)$.

In particular, the second part means that a function $f \in L^q$ can be written as a sum of a function in L^q and a function in L^r, hence the additive notation.

Proof of (1). Take a function $f \in L^p \cap L^r$; we will show $f \in L^q$. By the inequalities, we have $\frac{q}{p} > 1$ and $\frac{q}{r} < 1$, so there exists $0 < \alpha < 1$ satisfying

$$\frac{\alpha q}{p} + \frac{(1-\alpha)q}{r} = 1.$$

We would like to show the L^q-norm of f is finite, so we decompose it as follows:

$$\|f\|_q^q = \int_X |f|^q \,\mathrm{d}\mu = \int_X |f|^{\alpha q}|f|^{(1-\alpha)q} \,\mathrm{d}\mu = \||f|^{\alpha q}|f|^{(1-\alpha)q}\|_1,$$

which we can use in the LHS of Hölder's inequality with functions $|f|^{\alpha q}$, $|f|^{(1-\alpha)q}$ and exponents $\frac{p}{\alpha q}$, $\frac{r}{(1-\alpha)q}$ to yield

$$\||f|^{\alpha q}|f|^{(1-\alpha)q}\|_1 \leq \||f|^{\alpha q}\|_{p/\alpha q}\||f|^{(1-\alpha)q}\|_{r/(1-\alpha)q}.$$

Expanding the norms on the RHS gives us

$$\begin{aligned}
\||f|^{\alpha q}\|_{p/\alpha q} &= \left[\int_X (|f|^{\alpha q})^{\frac{p}{\alpha q}}\right]^{\frac{\alpha q}{p}} &&= \left[\int_X |f|^p\right]^{\frac{\alpha q}{p}} &&= \|f\|_p^{\alpha q} \\
\||f|^{(1-\alpha)q}\|_{r/(1-\alpha)q} &= \left[\int_X \left(|f|^{(1-\alpha)q}\right)^{\frac{r}{(1-\alpha)q}}\right]^{\frac{(1-\alpha)q}{r}} &&= \left[\int_X |f|^r\right]^{\frac{(1-\alpha)q}{r}} &&= \|f\|_r^{(1-\alpha)q},
\end{aligned}$$

so all in all we have

$$\|f\|_q^q = \||f|^{\alpha q}|f|^{(1-\alpha)q}\|_1 \leq \||f|^{\alpha q}\|_{p/\alpha q}\||f|^{(1-\alpha)q}\|_{r/(1-\alpha)q} = \|f\|_p^{\alpha q}\|f\|_r^{(1-\alpha)q},$$

showing us that

$$\|f\|_q^q \leq \|f\|_p^{\alpha q}\|f\|_r^{(1-\alpha)q} < \infty,$$

since the L^p and L^r norms of f are finite by hypothesis. Therefore the L^q-norm of f is also finite, so $f \in L^q$. ■

Proof of (2). For the second inclusion, take a function $f \in L^q$; we will show $f = g + h$ for $g \in L^p, h \in L^r$. Define the set

$$E \coloneqq \{x : |f(x)| > 1\}.$$

Take the indicator functions χ_E and χ_{E^c}, and define $g \coloneqq f\chi_E$, $h \coloneqq f\chi_{E^c}$. Clearly $f = g + h$, since for all $x \in X$, exactly one of $\chi_E, \chi_{E^c} = 1$, and now it stands to show integrability. We have

$$|g|^p = |f|^p\chi_E^p = |f|^p\chi_E \leq |f|^q\chi_E,$$

since $q > p$, meaning that $\|g\|_p \leq \|f\|_q$ (just integrate the above equation and take powers), so $\|g\|_p < \infty$ and therefore $g \in L^p$. Likewise, we have

$$|h|^r = |f|^r\chi_{E^c}^r = |f|^r\chi_{E^c} \leq |f|^q\chi_{E^c},$$

since the only contributing f-values are less than 1 in absolute value, in which case a higher exponent $(r > q)$ *reduces* their absolute value more. Therefore, we have $\|h\|_r \leq \|f\|_q$, so $h \in L^r$. The conclusion follows immediately. ■

4.2 Interpolation theorems

As mentioned earlier, interpolation theorems allow us to pass information from operators on some L^p spaces to those on others. In particular, having seen that for $1 \leq p < q < r \leq \infty$, we have the inclusion $(L^p \cap L^r) \subset L^q \subset (L^p + L^r)$, a common question in the application of L^p spaces is whether a linear operator (a structure-preserving map on a vector space; here, a function operating on the function space L^p) that is bounded on both L^p and L^r will also be bounded on L^q. The answer to this question turns out to be yes, as shown by the following two theorems. The proofs of these theorems require topics too advanced to provide here, but we will attempt to dissect the intuition behind each for the reader.

4.2.1 The Riesz-Thorin interpolation theorem

The motivation for the Riesz-Thorin interpolation theorem stems naturally from Theorem 4.6, namely (2): applications of L^p spaces are often concerned with operators defined on $L^p + L^r$.

One example from Fourier analysis invokes the Riemann-Lebesgue lemma, which shows that the Fourier transform maps $L^1(\mathbb{R}^d)$ boundedly into $L^\infty(\mathbb{R}^d)$, and Plancherel's theorem, which shows that the Fourier transform maps $L^2(\mathbb{R}^d)$ boundedly into itself, to show that the Fourier transform $\mathcal{F}$ extends to $(L^1 + L^2)(\mathbb{R}^d)$ by setting $\mathcal{F}(f_1 + f_2) = \mathcal{F}_{L^1}(f_1) + \mathcal{F}_{L^2}(f_2)$.

However, these are effectively two versions of the same operator: $\mathcal{F}_{L^1} : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ and $\mathcal{F}_{L^2} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. These are identical in that they agree on the space $(L^1 \cap L^2)(\mathbb{R}^d)$, and we are justified in considering them *the same*.

Therefore we may desire to study operators mapping two domain spaces L^{p_1} and L^{r_1} to two target spaces L^{p_2} and L^{r_2}, which map $L^{p_1} + L^{r_1}$ to $L^{p_2} + L^{r_2}$; we may expect that these operators also operate on L^{q_0}, mapping it into L^{q_1} (using similar notation to Theorem 4.6 for clarity). The Riesz-Thorin interpolation theorem gives us this result when this operator is linear:

Theorem 4.7 (Riesz-Thorin)**.** *Suppose that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, with $p_0, p_1, q_0, q_1 \in [1, \infty]$. If $q_0 = q_1 = \infty$, suppose also that ν is semifinite. For $0 < t < 1$, define p_t and q_t by*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \text{and} \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Suppose T is a linear operator from $L^{p_0}(X, \mathcal{A}, \mu) + L^{p_1}(X, \mathcal{A}, \mu)$ into $L^{q_0}(Y, \mathcal{B}, \nu) + L^{q_1}(Y, \mathcal{B}, \nu)$ such that $\|Tf\|_{q_0} \leq M_0 \|f\|_{p_0}$ for $f \in L^{p_0}(X, \mathcal{A}, \mu)$, and $\|Tf\|_{q_1} \leq M_1 \|f\|_{p_1}$ for $f \in L^{p_1}(X, \mathcal{A}, \mu)$. Then $\|Tf\|_{q_1} \leq M_0^{1-t} M_1^t \|f\|_{p_t}$ for $f \in L^{p_t}(X, \mathcal{A}, \mu)$.

While this is quite the mouthful to unpack, the underlying intuition is that linear operators between sumsets of L^p spaces have bounded norms, and the theorem furthermore provides this bound.

This is often useful for finding results on the more complex L^p spaces; L^1, L^2, and L^∞ have quite simple structure, and so we would like to work with these spaces instead of, say, $L^{174.3}$. The Riesz-Thorin interpolation theorem allows us to prove theorems in some of these simple cases and interpolate between them to prove the theorems on all intermediate L^p spaces, considerably simplifying our work.

However, not every operator under study is linear; a result on nonlinear operators was given by Józef Marcinkiewicz and bears his name, which we will now examine.

4.2.2 The Marcinkiewicz interpolation theorem

We will need to define another advanced notion that we glossed over in this paper, weak L^p. There is a lot more to the theory of weak L^p, but what is necessary to be understood for the Marcinkiewicz interpolation theorem is as follows:

Definition 4.8. Given a measurable function f on a measure space $(X, \mathcal{A}, \mu)$, define its *distribution function* $\lambda_f : (0, \infty) \to [0, \infty]$ by

$$\lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\}).$$

Definition 4.9. If f is a measurable function on a measure space $(X, \mathcal{A}, \mu)$ with $0 < p < \infty$, we define the *weak L^p norm* $[f]_p$ as

$$[f]_p = \left(\sup_{\alpha > 0} \alpha^p \lambda_f(\alpha) \right)^{\frac{1}{p}}.$$

Equivalently, the weak L^p norm is the best constant C in the inequality $\lambda_f(\alpha) \leq \frac{C^p}{\alpha^p}$ for all p. We define the space *weak* $L^p(X, \mathcal{A}, \mu)$ to be the set of all f where $[f]_p < \infty$.

Of note quickly is that the weak L^p norm is not actually a norm, since the triangle inequality fails, but it is often of use for generalizing some of the useful results of L^p spaces as we will shortly demonstrate. Now let T be a map from a vector space $\mathcal{V}$ of measurable functions on $(X, \mathcal{A}, \mu)$ to the space of all measurable functions on $(Y, \mathcal{B}, \nu)$, and we will need to bring in a few more definitions. Namely, Marcinkiewicz's interpolation theorem attempts to characterize bounds on *sublinear* maps, so we will need to define those:

Definition 4.10. T is *sublinear* if $|T(f + g)| \leq |Tf| + |Tg|$ and $|T(cf)| = c|Tf|$ for all $f, g \in \mathcal{V}$ and $c > 0$.

Contrast this with the definition of a *linear* operator, where the inequality is an equality. We are interested in two types of sublinear operators between L^p spaces, both of which have particular boundedness restrictions:

Definition 4.11. A sublinear map T is *strong type* (p, q), with $1 \leq p, q \leq \infty$, if $L^p(X, \mathcal{A}, \mu) \subset \mathcal{V}$, T maps $L^p(X, \mathcal{A}, \mu)$ into $L^q(Y, \mathcal{B}, \nu)$, and there exists a constant $C > 0$ such that $\|Tf\|_q \leq C\|f\|_p$ for all $f \in L^p(X, \mathcal{A}, \mu)$.

Strong type sublinear maps are bounded from $L^p \to L^q$: the L^q-norm of the image of $f \in L^p$ is at most a constant value times its L^p-norm. Likewise, we define notions of boundedness on operators between *weak* L^p spaces:

Definition 4.12. A sublinear map T is *weak type* (p, q), with $1 \leq p \leq \infty$ and $1 \leq q < \infty$, if $L^p(X, \mathcal{A}, \mu) \subset \mathcal{V}$, T maps $L^p(X, \mathcal{A}, \mu)$ into weak $L^q(Y, \mathcal{B}, \nu)$, and there exists $C > 0$ such that $[Tf]_q \leq C\|f\|_p$ for all $f \in L^p(X, \mathcal{A}, \mu)$. T is also weak type (p, ∞) if and only if T is strong type (p, ∞).

Weak type sublinear maps are bounded rather from $L^p \to$ *weak* L^q: the weak L^q-norm of the image of f is at most a constant value times its L^p-norm. We are now equipped to state and unpack the Marcinkiewicz interpolation theorem.

Theorem 4.13 (Marcinkiewicz). *Suppose that* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *are measure spaces,* $p_0, p_1, q_0, q_1 \in [1, \infty]$ *where* $p_0 \leq q_0, p_1 \leq q_1$*, and* $q_0 \neq q_1$*, and*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}, \qquad \text{where } 0 < t < 1.$$

If T *is a sublinear map from* $L^{p_0}(X, \mathcal{A}, \mu) + L^{p_1}(X, \mathcal{A}, \mu)$ *to the space of measurable functions on* Y *that is weak types* (p_0, q_0) *and* (p_1, q_1)*, then* T *is strong type* (p, q)*. More precisely, if* $[Tf]_{q_j} \leq C_j \|f\|_p$*, for* $j = 0, 1$*, then* $\|Tf\|_q \leq B_p \|f\|_p$ *where* B_p *depends only on* p_j, q_j, C_j *in addition to* p*; and for* $j = 0, 1$*,* $B_p|p - p_j|$ *(resp.* B_p*) remains bounded as* $p \to p_j$ *if* $p_j < \infty$ *(resp.* $p_j = \infty$*).*

A more digestible statement of the theorem is that if T is a bounded sublinear operator from L^p to weak L^p and from L^r to weak L^r, then T is also bounded from L^q to L^q for any $1 \leq p < q < r \leq \infty$, i.e. regular boundedness still holds even if T is only *weakly* bounded at the extremes.

Like Riesz-Thorin, Marcinkiewicz provides these bounds, but they are weaker estimates than those provided in Riesz-Thorin's theorem. Further comparing the two, Marcinkiewicz requires more stringent restrictions on p_j and q_j than Riesz-Thorin, and has weaker hypotheses: it considers sublinear operators rather than linear, and requires only weak boundedness at the extremes rather than strict boundedness. Therefore both see use in the application of L^p spaces, though these are rather too advanced to see exposition in this paper.

4.3 Further reading

The reader who wishes to learn more about the theory of L^p spaces may find Folland's *Real Analysis* instructive: it covers much of the content of this paper as well as some content this paper excludes, such as weak L^p and dual spaces. Stein and Shakarchi's *Functional Analysis* also includes quality content on L^p spaces and more about their nature as Banach spaces. As always, Google, the Math Stack Exchange, Math Overflow, and Wikipedia are useful sources as well.