PERIODIC TRAJECTORIES IN RATIONAL BILLIARDS

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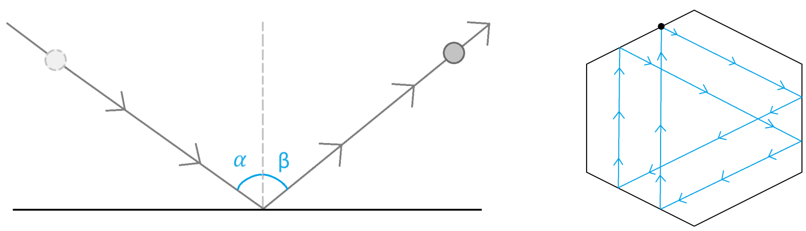
1. Introduction

Consider any billiard table that is a polygon. We can place a billiard ball anywhere on the table and push the ball in any direction. The ball will move in a straight line until it reaches the edge of the table. At this point, the ball will reflect off the edge and continue in a different direction such that the angle of incidence is equal to the angle of reflection as shown in Figure 1a. We will assume that once the ball is in motion, it will not stop moving. The path the ball takes is called its trajectory or orbit. A trajectory is periodic if the ball returns to its starting position and direction of motion periodically.



(A) (B)

Figure 1. (A) A billiard ball reflects off an edge of a billiard table such that the angle on incidence α and the angle of reflection β are equal. (B) A periodic trajectory in a hexagon.

Does every polygonal billiard table contain a periodic trajectory? Currently, this result has not been proven yet, but in 1986, Masur [1] proved a special case of this result, that is for rational polygons, and in 1998, Boshernitzan, Galperin, Krüger, and Troubetzkoy [2] proved a stronger version of this special case. In this paper, we will review the proof of this stronger theorem. A polygon is considered rational if all of its interior angles are rational multiples of π. It turns out there are special properties of rational polygons that can help us check for periodic trajectories. Given a starting position and direction of motion of the ball in a rational polygon, there are a finite number of directions the ball will move in its trajectory. Masur's theorem shows that for every billiard ball position and direction of motion in the polygon, there exists a billiard ball position along a periodic trajectory with an arbitrarily close direction of motion. This result can be used to prove the following stronger theorem.

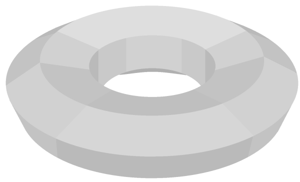
Theorem 1.1. [2] *Let Q be a rational polygon. There exists an arbitrarily close approximation to any billiard ball position and direction of motion in Q by a billiard ball position and direction of motion in a periodic trajectory in Q.*

2. Periodic trajectories in rational polygons

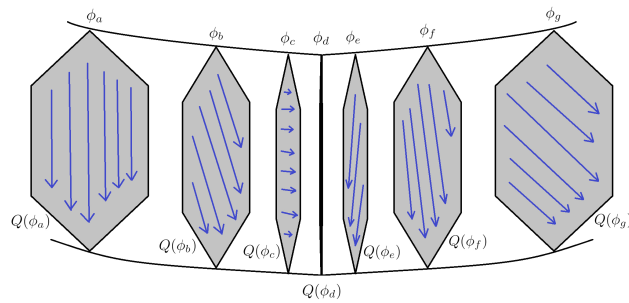
We start by defining a few terms that we will use throughout the rest of this paper.

Definition 2.1. Let Q be a rational polygon. The *phase space* M of Q is equal to $M = Q \times S^1$ where S^1 is the set of all unit vectors.

The phase space is the set of all elements of the form (q, ϕ) where q is any point in Q and ϕ is any direction in S^1. We can visualize the phase space of Q by taking a right prism and gluing together the two bases. An example of a phase space of a hexagon is shown in Figure 2a.



(A)



(B)

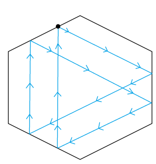
Figure 2. (A) The phase space of a hexagon is comprised of hexagonal cross-sections, each one representing a direction of motion of a billiard ball anywhere in the hexagon. (Adapted from V. M. Chapela and M. J. Percino, March 7, 2011; see [3].) (B) A few cross-sections which are floors of the phase space of the hexagon.

Each cross-section, called a *floor* of the phase space, represents a direction of motion of the billiard ball in Q (Figure 2b). We denote the floor with direction ϕ as $Q(\phi)$.

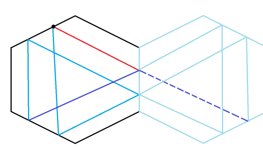
Definition 2.2. [4] The *flow* of any polygonal billiard table Q is defined as the set of all billiard trajectories in Q.

Definition 2.3. Define the *invariant surface* R_θ to be the set of floors that orbits containing the direction θ pass through where $\theta \in S^1$.

An important method that is used to prove Theorem 1.1 is called *unfolding*. We will look at an example of unfolding a periodic trajectory in the hexagon in Figure 3a. The *segments* of the trajectory are the straight line paths between reflections. Let e_1, e_2, e_3 and so on, be the set of edges in the hexagon the ball touches after crossing segments $1, 2, 3$ and so on, respectively. Then we can unfold the trajectory by first reflecting the hexagon and trajectory about e_1 (Figure 3b). We then reflect the new hexagon about e_2 (Figure 3c). If we continue this process, then we get an unfolded trajectory that is a straight line through a *corridor* of polygons (Figure 3d). This method can be used for any polygon.



(A)



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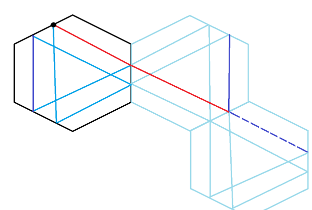
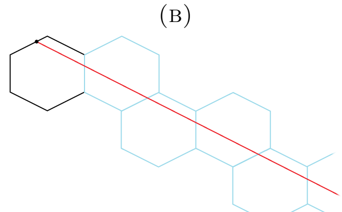


(C)



(D)

Figure 3. (A) A periodic trajectory in a hexagon. (B) After we reflect the hexagon and trajectory about e_1, the first segment and the reflected second segment form a straight line. (C) The first three segments of the trajectory unfold into a straight line. (D) The unfolded trajectory in a corridor of hexagons.

A *generalized diagonal* is any segment of an orbit, that starts at a vertex of the polygon and ends at a vertex, and has length equal to the the number of segments in the corresponding orbit. There are a countable number of generalized diagonals in R_θ. This is true since there are finitely many generalized diagonals of any given length and there are countably many possible lengths.

Definition 2.4. For any polygon Q, a billiard trajectory γ is *dense* in Q if for every point q and direction ϕ where (q, ϕ) is in the phase space, there exists an arbitrarily close approximation to q and ϕ by a point q' and direction ϕ' such that (q', ϕ') is in the trajectory γ.

Definition 2.5. For any polygon Q and any given $\varepsilon > 0$, a billiard trajectory γ is ε-*dense* in Q if for any point in Q, there exists a point on the trajectory γ such that the two points are at most ε apart.

Definition 2.6. A direction ϕ in S^1 is a *minimal direction* if the orbits of all points on the floor $Q(\phi)$ are dense in the polygon.

Definition 2.7. Define $Orb(x, \theta, N)$ as the section consisting of the first N segments of the forward orbit starting at point x and an initial direction of θ where θ is any direction in S^1, x is any point in R_θ, and $N \geq 1$.

We can replace θ by $-\theta$ to denote the backward orbit of the previous N segments before x. Note that $Orb(x, \theta, N) = \emptyset$ if the orbit starting at x with direction θ reaches a vertex less than N segments after x and $Orb(x, -\theta, N) = \emptyset$ if the orbit stops at a vertex less than N segments before x.

We need the following important lemma before we look at the proof of Theorem 1.1.

Lemma 2.8. *Let θ in S^1 be a minimal direction. Then for any $\varepsilon > 0$, there exists an integer N such that for all x in R_θ, either $Orb(x, \theta, N)$ is ε-dense in R_θ or $Orb(x, \theta, N) = \emptyset$.*

We refer to [2] for the proof of this lemma. This lemma also holds for $-\theta$ as well. Now we will review the proof of Theorem 1.1.

Proof of Theorem 1.1. Since minimal directions are dense in S^1 [5], there is an arbitrarily close approximation to any direction ϕ by a minimal direction θ_0. To prove the theorem, we must show that there is an arbitrarily close approximation to any point q_0 and minimal direction θ_0 by a point q_1 and direction θ_1 such that (q_1, θ_1) is on a periodic trajectory. To show this, pick any minimal direction θ_0. Then by Masur's theorem, we can find a periodic trajectory γ_1 containing a direction θ_1, that is arbitrarily close to θ_0. We then use the property that there is a corridor of N polygons consisting of both a section of the unfolded trajectory γ_0 with direction θ_0 and a section of the unfolded trajectory γ_1 with direction θ_1. This property can be proven by contradiction. Next, if we choose θ_1 so that the point p_0, the farthest point on the unfolding of γ_0 in the Nth polygon, and the point p_1, the farthest point on the unfolding of γ_1 in the Nth polygon, are at most $\varepsilon/2$ apart, then since the first N segments of γ_0 are $\varepsilon/2$-dense in R_{θ_0} by Lemma 2.8, the first N segments of the periodic trajectory γ_1 are ε-dense in R_{θ_1}. Given that this holds for any minimal direction ϕ_0, $\varepsilon > 0$, and N, we can find a point q_1 that is on the periodic trajectory γ_1 and is an arbitrarily close approximation to q_0 as desired. This implies that for any point and direction in the polygon, we can find an arbitrarily close approximation by a point and direction in a periodic trajectory. □

We refer to [2] for more details of this proof.

References

[1] Masur, Howard. "Closed trajectories for quadratic differentials with an application to billiards." (1986): 307-314.

[2] Boshernitzan, M., G. Galperin, T. Krüger, and S. Troubetzkoy. "Periodic billiard orbits are dense in rational polygons." Transactions of the American Mathematical Society 350, no. 9 (1998): 3523-3535.

[3] Chapela, V. M., and M. J. Percino "Twisted Polygonal Torus" see http://demonstrations.wolfram.com/TwistedPolygonalTorus, Wolfram Demonstrations Project, March 7 2011

[4] Park, S. Woo. "An introduction to dynamical billiards." see https://math.uchicago.edu/may/REU2014/REUPapers/Park.pdf (2014).

[5] Gutkin, Eugene. "Billiards in polygons." Physica D: Nonlinear Phenomena 19, no. 3 (1986): 311-333.