RATIONAL BILLIARDS

ADVAITH MOPURI

1. Introduction

Most things in the real world are quite chaotic, but contrastingly, much of math follows a rigid order. A mathematical adaption of the game of billiards, however, provides us with a good system in which to study both order in the form of periodic orbits, and chaos in the form of ergodic orbits.

Definition 1.1 (Billiards)**.** In the context of this paper, *billiards* refers to a game on a table with a smooth, closed, and infinitely thin surface. This surface is known as the *table*, and is smooth in order to ensure that the ball never loses speed, even after colliding with a wall. There is only 1 ball which is approximated as an infinitesimally small particle, and obeys the law of reflection.

Theorem 1.2 (Law of Reflection)**.** *The angle of incidence of a particle on a smooth surface is equal to the angle of reflection.*

The game begins with the ball moving with some velocity from some starting point inside the table. Throughout this paper, we will further study the path taken by a billiards ball when launched from some position with some velocity inside a billiards table. This path is known as an orbit.

Definition 1.3 (Orbit)**.** The *orbit* of a billiards ball is the set of all points inside the table which the ball passes through on the path due to its starting position and velocity.

The *period* of a periodic orbit is the number of times the billiard ball must hit the boundary of the table before returning to its initial state.

The orbits of most interest to us are those which are periodic and those which are ergodic (to be defined later), so we will study those first.

2. Periodic Orbits

Definition 2.1 (Periodic Orbits)**.** We call an orbit of the billiards ball periodic if it eventually returns to its starting point with the same direction that it started with.

2.1. Rectangles.

In a rectangle, the question of finding periodic trajectories is greatly simplified with the use of reflections.

Instead of the ball bouncing off of the walls of the table as usual, we can imagine it passing through the walls as if they were just air. Rather than being contained inside the table, the ball can now escape outside. To keep the ball contained though, we can simply reflect our rectangular table over the side which the ball passed through. The images below provide an example of this. Note that each position in the newly created rectangle corresponds to

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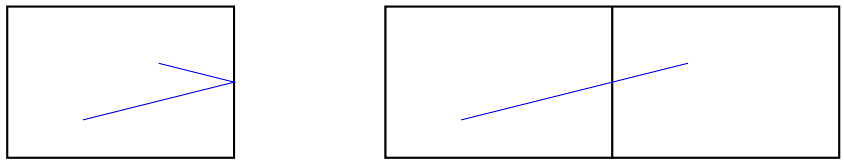

Figure 1. An example of reflecting a rectangle.

exactly one position in the original rectangle. So, we can track the complicated movement of the ball in the single original rectangular table by simply reflecting the table each time the ball hits the wall. Thus, we have turned the original bouncing movement into just a straight line.

Now that we have a method to simplify the movement of the billiards ball inside a rectangle, we can use it to address the question of periodic orbits. Let our rectangle have length l and height h. Then, if the original position of our ball is (x, y) inside the rectangle, the motion is periodic if we eventually end up at a position (x', y') in the "reflection space" such that $x \equiv x' \bmod l$ and $y \equiv y' \bmod h$.

One interesting case of this kind of motion is when the rectangular table has rational side lengths, namely, $l, h \in \mathbb{Q}$. Then, for any rational slope that the ball moves with, its orbit will always be periodic. To see this, let our slope be s. Then, we wish to show that there exist integers a and b for which $ks = a\frac{h}{l}$. However, such k and a clearly exist since s and $\frac{h}{l}$ are both rational.

We can even go one step further and compute how many times the ball hits a wall before returning to its starting point. Rewrite $\frac{k}{a}$ as $\frac{k'}{a'}$, where the latter fraction is in lowest terms. Then, it the ball starts anywhere but a corner of the table, it hits the walls $k' + a'$ times. If it starts at a corner, it hits the walls $k' + a' - 1$ times, because the last bounce on a corner hits only 1 wall rather than the usual 2.

Interestingly, if our slope s is not rational, then the orbit of the billiards ball is dense inside the rectanglular table.

2.2. **More Reflections.** In fact, all figures that tile the plane with reflections have periodic orbits due to similar reasoning as with rectangles. And, given that a shape has a periodic orbit, it must have infinitely many such orbits as infinitely many rational slopes exist.

What is more interesting, however, is that this method of reflections can be applied to all rational polygons.

Definition 2.2 (Rational Polygons)**.** A *rational polygon* is one with all its angles being rational multiples of $180°$ radians.

The explicit construction for a reflections pace for such a rational polygon is given by the Katok-Zemlyakov construction [Kat05].

The notion of *illumination* is also interesting in the case of rational polygons.

Definition 2.3 (Illumination)**.** Two points are said to *illuminate* each other if there exists a billiards trajectory containing both points.

Wolecki proved in [Wol24] that in any rational polygon, there are only finitely many pairs of points that illumintae each other.

2.3. **Triangles.** We have shown above that since equilateral triangles tile the plane with reflections and therefore have a periodic orbit. However, it turns out that all acute triangles have a periodic orbit, known as the Fagano orbit.

Theorem 2.4 (Fagano Orbit)**.** *In acute triangle ABC, let the foot of the altitudes to sides BC, AC, and AB be D, E, and F, respectively. Then, triangle DEF is a periodic trajectory.*

Proof. Consider any acute triangle. Then, we claim that its orthic triangle is a valid periodic orbit. To see this, consider the angles formed by the sides of the orthic triangle. It is well

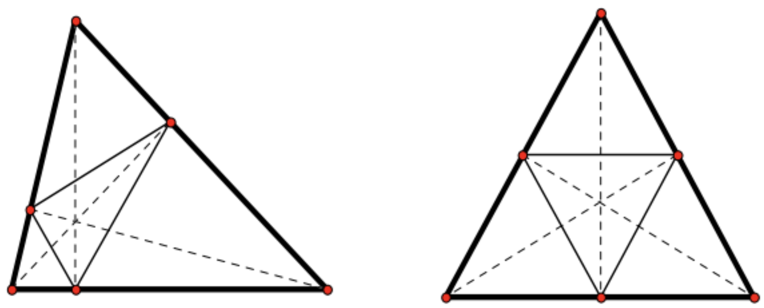

Figure 2. Examples of the Fagano orbit

known that the altitudes of the larger triangle are angle bisectors of the orthic triangle, so the angle of incidence must be the same as the angle of reflection, as required by 1.2.

Thus, the Fagano orbit is a valid periodic orbit. Note that this orbit does not work for obtuse triangles, as one of the vertices of the orthic triangle would lie outside the larger triangle. ■

Additionally, all triangles with rational angle measures (measured in degrees) also have periodic orbits. This is a direct consequence of the aforementioned Katok-Zemlyakov construction referenced from [Kat05].

3. Ergodic Orbits

In order to define what an ergodic orbit is, we must first define the measure to be applied on our 2-dimensional billiards table.

Definition 3.1 (2-dimensional Lebesgue Measure)**.** The 2-dimensional Lebesgue measure, λ_2, is defined as the unique measure that satisfies

$$\lambda_2\left((a_1, b_1] \times (a_2, b_2]\right) = (b_1 - a_1)(b_2 - a_2)$$

for $-\infty < a_1 < b_1 < \infty$, and similar for a_2 and b_2.

For the rest of the paper, it will be implicitly assumed that this measure is the one being used.

Definition 3.2 (Ergodic Orbits)**.** A trajectory is ergodic if the set of points not covered by the trajectory has measure 0.

Sinai famously proved that the following billiards table is ergodic [Sin63].
Interestingly, this billiard is chaotic as well.

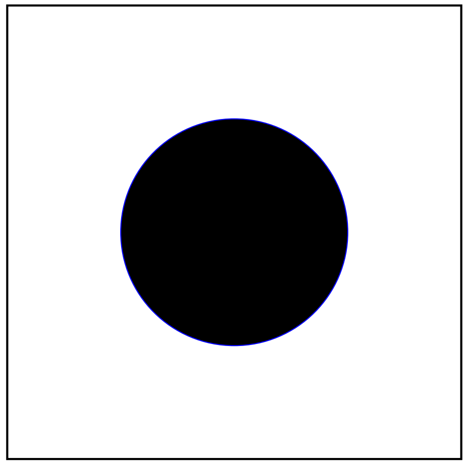

Figure 3. The Sinai billiard.

Definition 3.3 (Chaotic Billiards)**.** A billiards table is said to be *chaotic* if a slight change in the initial state of the ball (i.e. its position or velocity) causes a much larger change in the orbit.

For an example of a chaotic table (other than the Sinai billiard), consider the following pinball machine from [CM06].
Intuitively, it makes sense that a slight deviation in the initial state of a billiards ball on

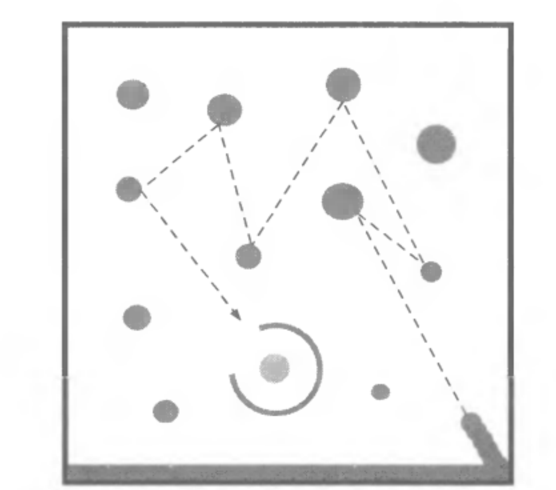

Figure 4. A chaotic pinball machine.

this table should cause a large change in the ball's trajectory. The many bouncers cause

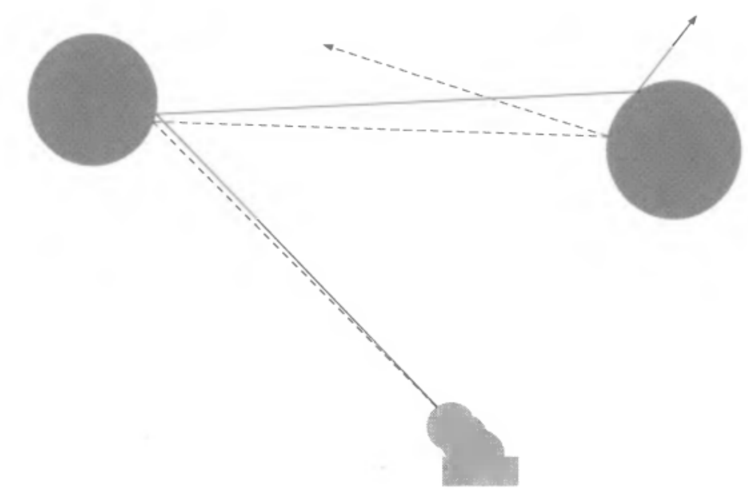

Figure 5. Chaos in the pinball machine.

unpredictable and chaotic motion. For example, consider the following 2 trajectories. Even the slightest change ends up having a huge effect on the trajectory of the ball.

4. Circles

The final interesting case that we will cover in this paper is billiards in a circular table.

In a circular table, 1.2 guarantees that every collision rotates the trajectory of the ball by a fixed angle ϕ. If ϕ is irrational, this means that the rotation is ergodic with respect to the previously defined Lebesgue measure. In fact, the rotation is uniquely ergodic, so the invariant measure of the rotation is unique.

The most interesting thing about circular billiards is the formation of caustics. For example, consider the image below. The caustic on this circular table is the central ring that

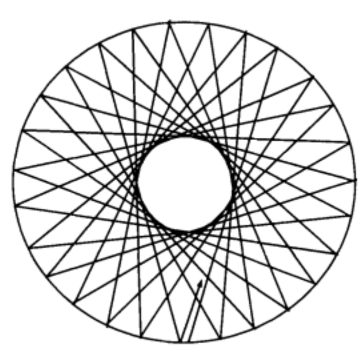

Figure 6. A caustic ring in a circular billiards table.

has no points of the ball's trajectory inside of it. However, the ball's trajectory is dense on the boundary of this caustic. It is impossible for the trajectory to be dense on the entire circle, though [CM06].

References

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