NONDIFFEOMORPHIC MANIFOLDS HOMEOMORPHIC TO 4-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. We give an overview of the methods by which one can prove the uniqueness of smooth structures on RR^n for $n \neq 4$, and then provide some interesting results in $\mathbb{R}^4$.

1. Introduction

All Euclidean spaces admit only one differentiable structure – except for $\mathbb{R}^4$. The first examples were found in 1982 by Michael Freedman and others, but it was a shock to individuals to discover that 4-dimensional Euclidean space was so strange while the other dimensions were so nice.

Since then, a variety of results have been proven: there is a continuum of non-diffeomorphic differentiable structures on $\mathbb{R}^4$, but this is a brief overview of the lower-dimensional and higher-dimensional cases, as well as some motivation for why they fail in 4-dimensions.

2. Lower Dimensions

For $n \leq 3$, we have that there is a unique smooth structure on $\mathbb{R}^n$. However, this happens for a few different reasons.

2.1. Dimension 1.

In one dimension, there is only one smooth structure (e.g. there is only one smooth structure on the real number line). Therefore, there are no exotic structures.

2.2. Dimension 2.

The proof of the uniqueness of a smooth structure on $\mathbb{R}^2$ can take many forms. Primarily, this utilizes the fact that piecewise-linear structure and smooth structure in dimensions less than or equal to 4 are identical.

Definition 2.1 (Piecewise linear manifold)**.** A piecewise linear (PL) manifold is a topological manifold with a piecewise linear structure. You can define it in terms of atlases much as you can a smooth manifold, but instead of smooth mappings, the mappings are constructed with piecewise linear functions.

One proves that there is only one PL structure on $\mathbb{R}^2$ with triangulations.

2.3. Dimension 3.

There is, in fact, a *unique smooth structure* in $\mathbb{R}^3$. The approach is much the same as dimension 2 - define a triangulation, and prove that the PL structures are identical, therefore the smooth structures are identical.

Smooth equivalence is the same as PL equivalence in small dimensions, so this works.

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3. The h-Cobordism Theorem

Definition 3.1 (Cobordisms)**.** A *cobordism* between two oriented manifolds M, N of dimension n is any such oriented manifold W of dimension $(n+1)$ with boundary

$$\partial W = \overline{M} \cup N.$$

Here $\overline{M}$ denotes M with the opposite orientation.

Why do we flip the orientation of M? Take the manifold $W = M \times [0,1]$ as an example. We have that $\partial W = \overline{M} \times 0 \cup M \times 1$, and interestingly this ties into the fundamental theorem of calculus resulting in $f(b) - f(b)$ **(go into more detail!)**

Definition 3.2 (h-Cobordisms)**.** A *h-Cobordism* is a cobordism which is homotopically equivalent to the **trivial cobordism** $M \times [0,1]$.

Then, we get the following result: that a homotopically-trivial cobordism is in fact also smoothly-trivial.

Theorem 3.3 (h-Cobordism Theorem)**.** *Let M^n and N^n be simply connected oriented manifolds of dimension n that are h-cobordant through the $(n+1)$ dimensional manifold W^{n+1}. Then, if $n \geq 5$, there is a diffeomorphism*

$$W \equiv M \times [0,1]$$

which can be chosen to be the identity from $M \subset W$ to $M \times 0 \subset M \times [0,1]$. M, N must be diffeomorphic.

The proof of this theorem is long – we will go over a brief outline, which involves translating the fact that the homology group of W is zero into a more topological interpretation involving a handle decomposition.

The way this is done is to first choose a Morse function on W that has distinct critical points. This lets you construct a handlebody decomposition of W, which gives a formula for converting M into N via handle addition.

Then, with a variety of techniques, you can 'cancel' all the handles, which leaves W diffeomorphic to the trivial $h-$cobordism, which implies that M, N are diffeomorphic.

Definition 3.4 (ρ-cobordism)**.** Take a Morse function $f : W \to [0,1]$.

We define the ρ-cobordism as

$$W_p = f^{-1}[0, \rho)$$

where $\rho \in [0,1]$.

As ρ passes some critical value of the Morse function f, then the topology of W changes. Essentially, the topology of $W_{\rho-\epsilon}$ and $W_{\rho+\epsilon}$ differs by exactly one handle attachment. Passing a critical point of degree k is the same as attaching a thickened $k-$disk handle.

3.1. Handle Decomposition.

For each pair of handles with the boundary intersecting with a $k-1$ handle, we can cancel the handle by deleting intersection points with opposite signs (insert figure here). Then, because this cancels all the handles, this gives us a handle-less decomposition of W. Therefore, this is the trivial cobordism, and M, N are diffeomorphic.

3.2. **The Whitney Trick.** We can consider two submanifolds P^k and Q^{n-k} of the manifold M^m. Note that this is not an issue when they are of complementary dimensions, as they intersect at some finite number of points, and getting rid of them can be reduced to getting rid of pairs of intersection points with opposite signs.

Therefore, we have to consider when P, Q are the attaching/belt spheres of handles. For any two intersection points of opposite signs, we can choose a path that connects them wholly in P and another wholly in Q. These two paths bound a circle in neither P nor Q.

It is known that embeddings are dense in the space of mappings from $A^n \to B^{2n+1}$. Therefore, immersions of disks in manifolds of dimension at least 5 can always be approximated by embeddings. Therefore, this *Whitney disk* can be approximated by embeddings in this case.

Using this Whitney disk as a guide (and some additional machinery) we can make the intersection points 'disappear' by pushing P past Q.

3.3. **Homology Recovery.** We can translate the handle decomposition into a chain complex

$$C_k = \mathbb{Z}\{k - \text{handles } h^k_\alpha\}$$

and boundary maps $\partial_k : C_k \to C_{k-1}$ given by

$$\partial_k(h^k_\alpha) = \sum \langle h^k_\alpha \mid h^{k-1}_\beta \rangle \cdot h^{k-1}_\beta$$

where $\langle h^k_\alpha \mid h^{k-1}_\beta \rangle$ is the incidence number of the two handles, which is just the handle intersection number.

The resulting homology groups

$$H_k(C_*) = \text{Ker } \partial_k / \text{Im } \partial_{k+1}$$

are just the relative homology groups of W, M.

Proof. Take a handle decomposition of W. Then, we can represent all of the boundary operators ∂_k as matrices with only 1s along the diagonal via some combination of row operations and orientation changes, because the relative homology of W, M is the 0 group. Since $\partial_k \partial_{k+1} = 0$, then for every k-handle h^k_a there exists either a $k+1$ or $k-1$ handle such that the boundary of the original handle is that handle. ∎

Therefore, there are no exotic structures in $\mathbb{R}^n$ for $n \geq 5$.

4. 4-Dimensions

Primarily, the Whitney trick fails in 4-dimensions because there is no guarantee that the embeddings are dense for dimensions smaller than 5. There is no 'room' to twist the Whitney disk into such that the handle can be removed.

Theorem 4.1. *There are uncountably many exotic differentiable structures on* $\mathbb{R}^4$.

The proof is extremely involved and uses gauge theory heavily, but it is an important result that should be noted.

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