SPHERE EVERSION

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ABSTRACT. In this expository paper, we begin by using winding numbers and covering spaces to rigorously prove it impossible to evert a circle. Then, we look at a sketch of a proof for sphere eversion using the turning number. Finally, we analyze the Bednorz-Bednorz sphere eversion and model it in the 2D graphing calculator Desmos.

Introduction

Over time, the field of math has faced a challenge: as our collective knowledge has increased, the complexity of questions we have posed has also increased. Ancient mathematicians like Eratosthenes and Pythagoras could simply say they were calculating the circumference of the Earth, or had found a relationship between the side lengths of right triangles. Modern mathematicians, on the other hand, need to explain formidable problems like the Riemann Hypothesis or the Poincaré Conjecture, for which even a basic grasp requires solid mathematical foundation.

However, amidst all this complexity, we occasionally encounter beautiful problems—ones that not only challenge us, but are also deceptively simple to explain. One example is the problem of sphere eversion: Can you turn a sphere inside-out under the following rules?

- The sphere can be bent, stretched, and shrunk in any way.
- The sphere can be pulled through itself.
- The sphere cannot be torn or glued together.
- The sphere cannot be "creased."

The first solution that may come to mind is to push two opposite ends of the sphere all the way through one another, revealing the sphere's insides (see Figure 1). However, this strategy introduces a sharp "crease" around the middle of the sphere, which violates the above rules. It turns out, the problem of sphere eversion is incredibly challenging. In fact, until the mid-20th century, mathematicians believed it impossible to evert a sphere.

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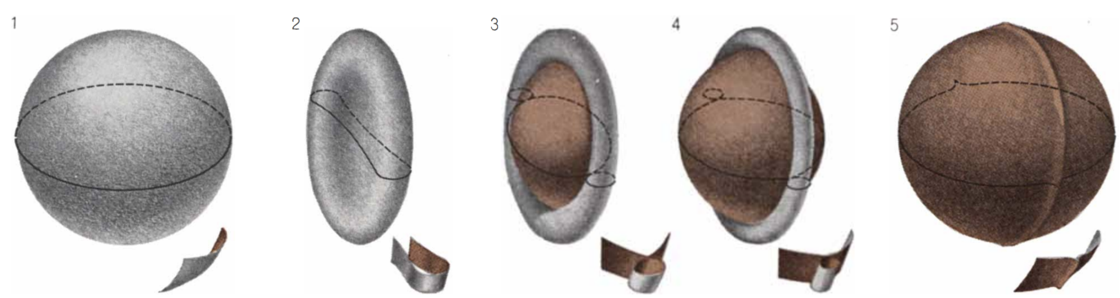

Figure 1. An example of an invalid sphere eversion.

Nevertheless, in 1957, Stephen Smale showed that "any two C^2 immersions of S^2 in E^3 are regularly homotopic" [1], which finally indirectly proved it possible to evert a sphere. Raoul Bott, his graduate advisor, famously told him that his claim was fundamentally incorrect, believing it impossible to evert a sphere. Bott was wrong, of course, and was later convinced that Smale's logic was valid, publishing *A Classification of Immersions of the Two-Sphere* in March 1959.

Still, while we knew that a sphere eversion existed, we had little idea what one looked like. It took another 4 years for the first explicit example of an eversion to be discovered by Arnold S. Shapiro in 1961. However, Shapiro did not publish this. It was only after Shapiro described this eversion to mathematician Bernard Morin (who, interestingly, was blind [2]) that the explicit eversion spread throughout the mathematical community. Eventually, an article published in *Scientific American* in 1966 [3] finally introduced the problem of sphere eversion, along with Shapiro and Morin's explicit eversion, to a wide audience.

1. Homotopies

The first step in proving sphere eversion is rigorously defining what it means to "evert" something. We begin by defining smooth functions in Euclidean space.

Definition 1.1. Let X be some subset of $\mathbb{R}^n$ and let $f : X \to \mathbb{R}^m$ be a function. We say that f is *smooth* if it has continuous partial derivatives of all orders, or if there exists an extended function F of f with continuous partial derivatives of all orders.

In differential topology, surfaces are not thought of as collections of points, but are defined in terms of smooth parametrized functions across Euclidean spaces. The sphere and the everted sphere are examples of this. Usually, their parameterizations are functions $\psi, \psi' : [-\pi, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}^3$ where

$$\psi(\theta, \phi) = (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), \sin(\phi))$$
$$\psi'(\theta, \phi) = (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi))$$

However, it is sometimes inconvenient or impossible to parametrize entire surfaces with a single mapping from Euclidean space. This leads us to the definition of manifolds, one of the most fundamental objects of topology.

Definition 1.2. A surface $X \in \mathbb{R}^N$ is an *n-dimensional manifold* (or simply an *n-manifold*) if it can be locally parameterized by smooth functions from $\mathbb{R}^n$. In other words, X is a manifold if and only if, for all $x \in X$, there is a smooth parameterization $\phi : U \to V$, where $U \subseteq \mathbb{R}^n$, $x \in V$, and $V \subseteq X$.

Now, we define homotopies.

Definition 1.3. Let $f_0, f_1 : X \to \mathbb{R}^m$ be smooth maps. We say that f_0 and f_1 are *homotopic* if there exists a smooth map $H : X \times [0, 1] \to \mathbb{R}^m$ such that $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$ for all $x \in X$. We call H a *homotopy* between f_0 and f_1.

We can imagine homotopies as "movies," with their additional dimension $[0, 1]$ thought of as time. In this analogy, a homotopy begins at time $t = 0$ as the initial map f_0, and then smoothly transitions into map f_1 by time $t = 1$.

Definition 1.4. An *immersion* is a differentiable function $f : X \to Y$ between manifolds X and Y, whose derivative is everywhere injective.

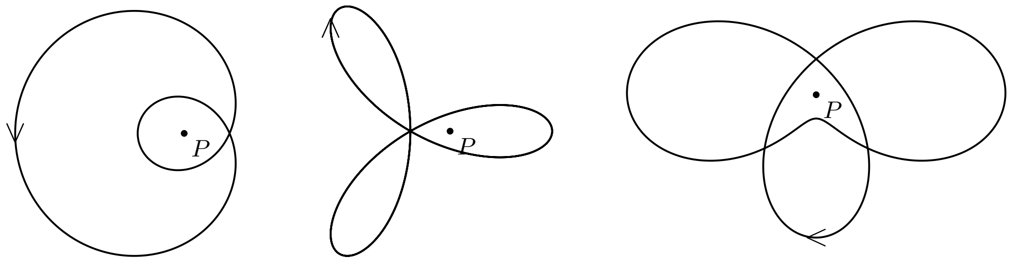

Figure 2. Curves with winding numbers 2, −2, and −1 around point P, respectively.

In other words, there should be no creasing, tearing, or gluing between manifolds X and Y in an immersion between X and Y.

Definition 1.5. A *regular homotopy* is a homotopy that remains an immersion of some manifold X throughout the homotopy. Two maps $f_0, f_1 : X \to \mathbb{R}^m$ are *regularly homotopic* if there exists a regular homotopy between them.

So, our goal is to show that the sphere and the everted sphere are regularly homotopic. We cannot crease, cut or glue the sphere anytime during the homotopy because that would prevent it from being smooth and regular. However, we are still allowed to pull the sphere through itself, bend it, stretch it, and shrink it in any way because the homotopy can still remain smooth and regular.

2. Winding Numbers

Before proving sphere eversion, it is helpful to first solve the problem of circle eversion. Note that, unlike the sphere, the circle cannot be everted. We begin by defining curves, which are a special kind of 1-manifold.

Definition 2.1. A *curve* is a smooth mapping $\gamma : [0, 1] \to \mathbb{C}$.

It is usually more convenient to work with complex numbers in the complex plane, but curves may also map to $\mathbb{R}^2$ instead of $\mathbb{C}$.

Definition 2.2. A *closed curve* is a curve that smoothly starts and ends at the same place. More formally, a curve γ is a closed curve if $\gamma^{(k)}(0) = \gamma^{(k)}(1)$ for all $k \in \mathbb{Z}_{\geq 0}$, where $f^{(n)}$ denotes the nth derivative of f.

Because curves are defined as mappings from the real number line, all curves come with direction. To indicate this direction, curves are usually drawn with an arrow, as shown in Figure 2.

The circle is defined as the curve $\gamma(t) = e^{2\pi ti}$, and the everted circle as the curve $\gamma'(t) = e^{-2\pi ti}$. One strategy commonly used to prove that the circle γ and the everted circle γ' are not homotopic is to find some characteristic of the two that is invariant under homotopies, and show that this characteristic is different for the two curves. Such a characteristic is called a "homotopy invariant."

Remark 2.3. Note that the regular homotopy is just a stronger version of the homotopy. This means that by showing that the circle γ and the everted circle γ' are not homotopic, we are also showing that they are not regularly homotopic.

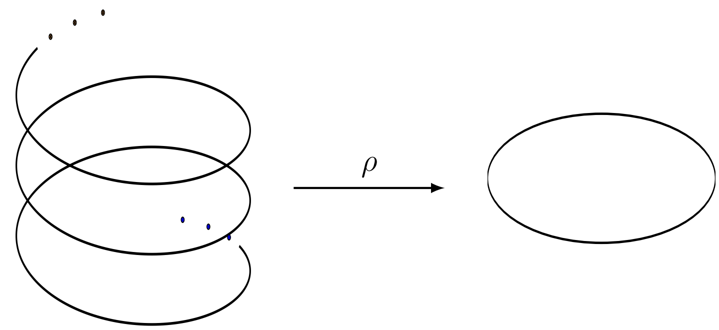

Figure 3. An example of a helix-shaped covering space of the unit circle $\mathbb{S}^1$, with the covering map $\rho : \mathbb{R} \to \mathbb{S}^1$, where $\rho(t) = e^{2\pi t}$.

In this paper, we will use a homotopy invariant called the winding number. Informally, the winding number is the number of times a curve "winds" counter-clockwise around some external point. Imagining a person fixed at a point, the winding number of a curve around that point would be the net number of times the person would have to fully turn around counter-clockwise when following the curve from start to end. Some examples are provided in Figure 2.

Remark 2.4. Another commonly used homotopy invariant for circle eversion is the *turning number*. The turning number is roughly defined as the number of times the normal vector (or the tangent vector) of a curve rotates counter-clockwise as it smoothly travels along the curve. This is equal to the total curvature of the curve divided by 2π.

It turns out that rigorously defining the winding number is far from trivial. Intuitively, the winding number is equal to the measure of the angle formed by $\gamma(0)$, $\gamma(1)$, and point P divided by 2π. However, it is difficult to differentiate between coterminal angles with algebraic functions, so this approach is not simple. In this paper, we will do this using structures called "covering spaces." To define covering spaces, we must also define homeomorphisms.

Definition 2.5. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, and let $f : X \to Y$ be a map between the two spaces. We say that f is a *homeomorphism* if it is bijective, continuous, and if its inverse function $f^{-1} : Y \to X$ is also continuous. If there exists a homeomorphism $f : X \to Y$, then we say that X and Y are *homeomorphic*.

Definition 2.6. For a space X, let $\rho : \tilde{X} \to X$ be a mapping such that for all points $x \in X$, x has an open neighborhood $U \subset X$ where $\rho^{-1}(U)$ is a union of disjointed sets that are all homeomorphic to U through ρ. ρ is called a *covering map* of X, and $\tilde{X}$ is called a *covering space* of X.

An example of a covering map for the unit circle $\mathbb{S}^1$ is provided in Figure 3. In order to apply covering maps in a useful way, we first need the following theorem.

Theorem 2.7. *Let $\rho : \tilde{X} \to X$ be a covering map. Given continuous maps $\gamma : Y \times [0,1] \to X$ and $\tilde{\gamma}_0 : Y \times \{0\} \to \tilde{X}$ such that $\rho \circ \tilde{\gamma}_0 = \gamma|_{Y\times\{0\}}$, there exists a unique and continuous map $\tilde{\gamma} : Y \times [0,1] \to \tilde{X}$ such that $\rho \circ \tilde{\gamma} = \gamma$. $\tilde{\gamma}$ is called the "lifted path of γ through ρ."*

Proof. While we will not prove this theorem, a relatively quick proof is provided in [4, pg.30–31]. Additionally, while it does appear complex at first, our use of this theorem should be fairly intuitive. ∎

Theorem 2.8. *Any curve $\gamma : [0,1] \to \mathbb{C}\backslash P$ can be represented with unique and continuous polar functions $r : [0,1] \to \mathbb{R}_{>0}$ and $\theta : [0,1] \to \mathbb{R}$ such that $\gamma(t) = P + r(t)e^{i\theta(t)}$ for all $t \in [0,1]$.*

Proof. We first show that the map $\rho : \mathbb{R}_{>0} \times \mathbb{R} \to \mathbb{C}\backslash P$ where $\rho(r,\theta) = P + re^{i\theta}$ is a covering map of $\mathbb{C}\backslash P$. Let the open neighborhoods U of $\mathbb{C}\backslash P$ be

$$\mathrm{Im}(z) > 0,\ \mathrm{Re}(z) < 0,\ \mathrm{Im}(z) < 0, \text{ and } \mathrm{Re}(z) > 0.$$

All points $z \in \mathbb{C}$ fall in at least one of these neighborhoods. The neighborhoods have the following inverses, respectively,

$$\bigcup_{n\in\mathbb{Z}} r > 0,\ \theta \in \left(n, n + \frac{\pi}{2}\right)$$

$$\bigcup_{n\in\mathbb{Z}} r > 0,\ \theta \in \left(n + \frac{\pi}{2}, n + \pi\right)$$

$$\bigcup_{n\in\mathbb{Z}} r > 0,\ \theta \in \left(n + \pi, n + \frac{3\pi}{2}\right)$$

$$\bigcup_{n\in\mathbb{Z}} r > 0,\ \theta \in \left(n + \frac{3\pi}{2}, n + 2\pi\right)$$

These inverses are all unions of disjoint open sets in $\tilde{X}$. Additionally, ρ is bijective when restricted to just these open sets, and the mappings ρ and ρ^{-1} are continuous. So, the above sets are all homeomorphic to their open neighborhoods in X, which means ρ is a covering map of $\mathbb{C}\backslash P$.

Let $\gamma : [0,1] \to \mathbb{C}\backslash P$ be a curve. Let Y be a single point $\{0\}$, and let $\tilde{\gamma}_0 : \{0\} \times \{0\} \to \mathbb{R}_{>0} \times \mathbb{R}$ be the continuous function $\tilde{\gamma}_0(y,t) = (|\gamma(t)|, \arg(\gamma(t)))$. If X is $\mathbb{C}\backslash P$ and $\tilde{X}$ is $\mathbb{R}_{>0} \times \mathbb{R}$, then by Theorem 2.7, there must exist a unique continuous $\tilde{\gamma}$ such that $\gamma = \rho \circ \tilde{\gamma}$. Let the component functions for $\tilde{\gamma}$ be continuous functions $r : [0,1] \to \mathbb{R}_{>0}$ and $\theta : [0,1] \to \mathbb{R}$. Because $\tilde{\gamma}$ is lifted through ρ, we know that $\gamma(t) = P + r(t)e^{i\theta(t)}$ for all $t \in [0,1]$. Therefore, we know that r and θ are valid polar functions that exist, are unique, and are continuous for any curve γ. ■

Now, we can finally provide a rigorous definition of the winding number.

Definition 2.9. Let $\gamma : [0,1] \to \mathbb{C}\backslash P$ be a curve with external point P. From Theorem 2.8, let the corresponding polar function of γ be r and θ. The *winding number* $W(\gamma, P)$ of γ around point P is equal to

$$\frac{\theta(1) - \theta(0)}{2\pi}.$$

Remark 2.10. The winding number is also defined in complex analysis as

$$W(\gamma, P) = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)}{\gamma(t) - P} dt,$$

or in differential geometry as

$$W(\gamma, 0) = \frac{1}{2\pi} \int_\gamma \frac{x \cdot dy - y \cdot dx}{x^2 + y^2}.$$

In this paper, we use covering maps to define the winding number since they are rooted in topology. However, the homotopy invariance of the winding number can also be proven using other definitions of the winding number.

3. Circle Eversion

We now prove it impossible to evert a circle by proving that the winding number is a homotopy invariant. First, we need to prove a couple lemmas.

Lemma 3.1. *Any homotopy between two curves $H : [0,1] \times [0,1] \to \mathbb{C}\backslash P$ can always be represented with unique and continuous polar functions $r : [0,1] \times [0,1] \to \mathbb{R}_{>0}$ and $\theta : [0,1] \times [0,1] \to \mathbb{R}$ such that $H(y,t) = P + r(t)e^{i\theta(t)}$ for all $t \in [0,1]$.*

Proof. This lemma and its proof are extensions of Theorem 2.8. We still use covering space ρ, but instead of letting Y be a single point $\{0\}$, we let it be the unit interval $[0,1]$. This makes the initial lifted path $\tilde{H}_0$ be $\tilde{H}_0 : [0,1]\times\{0\} \to \mathbb{R}_{>0}\times\mathbb{R}$, which itself is a lifted curve and can be continuous by Theorem 2.8. This makes the lifted path of $H : [0,1]\times[0,1] \to \mathbb{C}\backslash P$ through ρ be the function $\tilde{H} : [0,1]\times[0,1] \to \mathbb{R}_{>0}\times\mathbb{R}$, which is unique and continuous by Theorem 2.8. $\tilde{H}$ has continuous component functions $r : [0,1]\times[0,1] \to \mathbb{R}_{>0}$ and $\theta : [0,1]\times[0,1] \to \mathbb{R}$ such that $H(y,t) = P + r(y,t)e^{i\theta(y,t)}$ for all $(y,t) \in [0,1] \times [0,1]$. Therefore, we once again know that r and θ are valid polar functions, and that they exist, are unique, and are continuous for any homotopy H between two curves. ■

Lemma 3.2. *The winding number $W(\gamma, P)$ of a closed curve γ is always an integer.*

Proof. In a closed curve, $\gamma(0) = \gamma(1)$, so $P + r(0)e^{i\theta(0)} = P + r(1)e^{i\theta(1)}$. This implies that $\theta(0)$ and $\theta(1)$ must be coterminal angles, which means $\theta(1) = \theta(0) + 2\pi k$ for some $k \in \mathbb{Z}$. Now, calculating the winding number gets

$$W(\gamma, P) = \frac{\theta(1) - \theta(0)}{2\pi} = \frac{(\theta(0) + 2\pi k) - \theta(0)}{2\pi} = \frac{2\pi k}{2\pi} = k.$$

Therefore, $W(\gamma, P) = k \in \mathbb{Z}$ for any closed curve γ. ■

Now, we are finally ready to show that the winding number is homotopy invariant.

Theorem 3.3. *The winding number of a closed curve $\gamma : [0,1] \to \mathbb{C}\backslash P$ at point P is homotopy invariant.*

Proof. Let $H : [0,1] \times [0,1] \to \mathbb{C}\backslash P$ be a homotopy between two curves. By Lemma 3.1, the polar function $\theta : [0,1] \times [0,1] \to \mathbb{R}$ of H is continuous. This means that the winding number $W(H,P,t) = \frac{\theta(1,t)-\theta(0,t)}{2\pi}$ must be continuous throughout the homotopy.

Because the initial curve γ is closed, the curve must remain closed throughout H, since H would otherwise not be smooth (this would be like cutting the curve, which is against the rules of the homotopy). So, by Lemma 3.2, the winding number of the curve must remain an integer throughout H.

Thus, since the winding number is both continuous and discrete, it must be constant throughout H. This implies that the winding numbers of the initial curve and the final curve of any homotopy must be equal. Thus, the winding number is homotopy invariant. ■

Theorem 3.4. *The circle $\gamma(t) = e^{2\pi ti}$ and the everted circle $\gamma'(t) = e^{-2\pi ti}$ are not homotopic.*

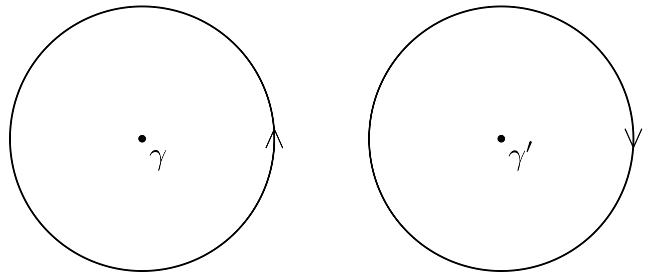

Figure 4. The circle $\gamma : [0,1] \to \mathbb{C}$ and the everted circle $\gamma' : [0,1] \to \mathbb{C}$, parameterized by maps $\gamma(t) = e^{2\pi t}$ and $\gamma'(t) = e^{-2\pi t}$.

Proof. By Theorem 2.8, the circle and the everted circle must be representable by unique polar functions r and θ. Let point P be the origin 0. For the circle, functions r and θ are $r(t) = 1$ and $\theta(t) = 2\pi t$ since $\gamma(t) = P + r(t)e^{i\theta(t)} = e^{2\pi ti}$. For the everted circle, functions r and θ are $r(t) = 1$ and $\theta(t) = -2\pi t$ since $\gamma'(t) = P + r(t)e^{i\theta(t)} = e^{-2\pi ti}$. So, calculating each of their winding numbers,

$$W(\gamma, 0) = \frac{\theta(1) - \theta(0)}{2\pi} = \frac{2\pi - 0}{2\pi} = 1$$
$$W(\gamma', 0) = \frac{\theta(1) - \theta(0)}{2\pi} = \frac{-2\pi - 0}{2\pi} = -1$$

Thus, because the winding numbers of the circle γ and the everted circle γ' are not equal, the two curves cannot be homotopic by Theorem 3.3. ■

This statement can be generalized to closed curves mapping to $\mathbb{C}$ rather than $\mathbb{C}\backslash P$ because any homotopy $H : [0,1] \times [0,1] \to \mathbb{C}$ can be transformed into some homotopy $H' : [0,1] \times [0,1] \to \mathbb{C}\backslash P$ by shifting the curve such that point P remains in its original region throughout the homotopy.

4. Turning Number

While sphere eversion is similar to circle eversion in many ways, it is much more difficult to prove. For this reason, the remainder of this paper will be less rigorous, pointing to other reputable sources. This section will instead provide a brief overview of the general method of indirectly proving sphere eversion using the turning number.

Intuitively, it is easier to prove something possible than to prove something impossible. This is because anything possible can be proven by existence, while proving anything impossible requires the consideration of (usually many) miscellaneous cases. However, the opposite is true in the case of circle and sphere eversion. It is much easier to prove that you *cannot* evert a circle than it is to prove that you *can* evert a sphere.

This is, in part, due to the complexity in visualizing a sphere eversion—both mentally and programmatically. Even if mid-20th century topologists could envision a relatively simple eversion, they certainly did not have the computing power necessary to model it. Additionally, a homotopy invariant, the key to proving circle eversion, can only prove that surfaces are *not* regularly homotopic. In order to prove that the sphere and the everted sphere are homotopic, we need a stronger version of the homotopy invariant.

Definition 4.1. A *complete homotopy invariant* is a homotopy invariant that, when constant between two surfaces, implies that they are homotopic.

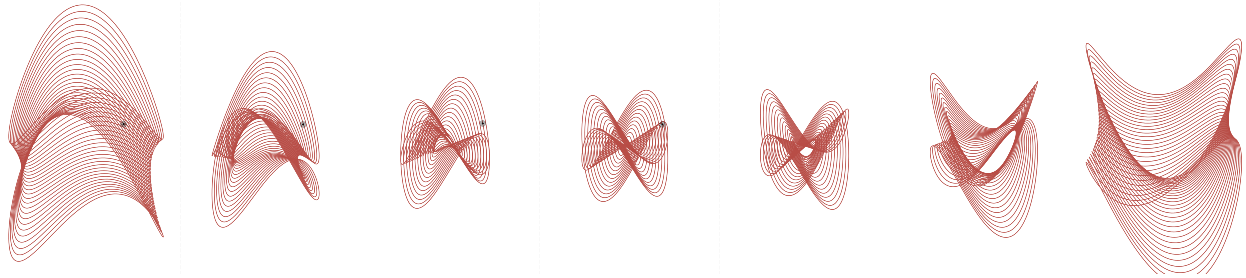

Figure 5. The Bednorz-Bednorz cylindrical eversion (for $n = 2$) modeled in Desmos with 3D contour lines. The Desmos graph is available at `https://www.desmos.com/calculator/evacdlgwgo`.

Remark 4.2. It turns out, the winding number is a complete homotopy invariant for curves. This follows closely from the Whitney-Graustein theorem, which is proven in [5]. However, this proof is not applicable to 3D surfaces, so we will use another complete homotopy invariant to prove sphere eversion.

The turning number is a complete homotopy invariant for all immersions of the sphere. A rough intuition for this can be found in [6]. The turning number of a closed surface is roughly defined as the number of "cups" and "bowls" in the surface minus the number of "saddles" in the surface. Because both the sphere and the everted sphere have one cup, one bowl, and no saddles, both surfaces have the same turning number and are therefore regularly homotopic.

5. Bednorz-Bednorz Eversion

Another method of proving sphere eversion is parameterizing an actual sphere eversion. This means finding the exact equations for a regular homotopy between the sphere and the everted sphere, showing that the homotopy begins with a sphere and ends with an everted sphere, and showing that the equations are smooth. This is possible with the recently discovered Bednorz-Bednorz sphere eversion [7], which can be parameterized with relatively simple equations. This not only makes the eversion much easier to describe and share, but also means that it can be more conveniently modeled on a computer. One example of this is in Ricky Reusser's website on sphere eversion [8], where the Bednorz-Bednorz eversion is modeled with detail and speed.

Remark 5.1. Figures 5 and 6 are modeled with the 2D graphing calculator Desmos. However, Desmos is not built to render in 3D—it cannot provide high quality 3D visuals, and cannot render these visuals quickly. In general, it is much more reasonable to use more sophisticated modeling software to properly visualize the eversion. The figures in this paper are only modeled with Desmos as a fun challenge and to demonstrate the simplicity of the equations of the Bednorz-Bednorz eversion.

The general idea for the Bednorz-Bednorz eversion is to first construct a cylindrical eversion, and then extend this to a sphere, similar to how a sphere can be topologically considered a cylinder with its upper and lower bases smoothly "capped off." We begin with the parameterizations for the cylindrical eversion. Note that this cylinder has infinite height and

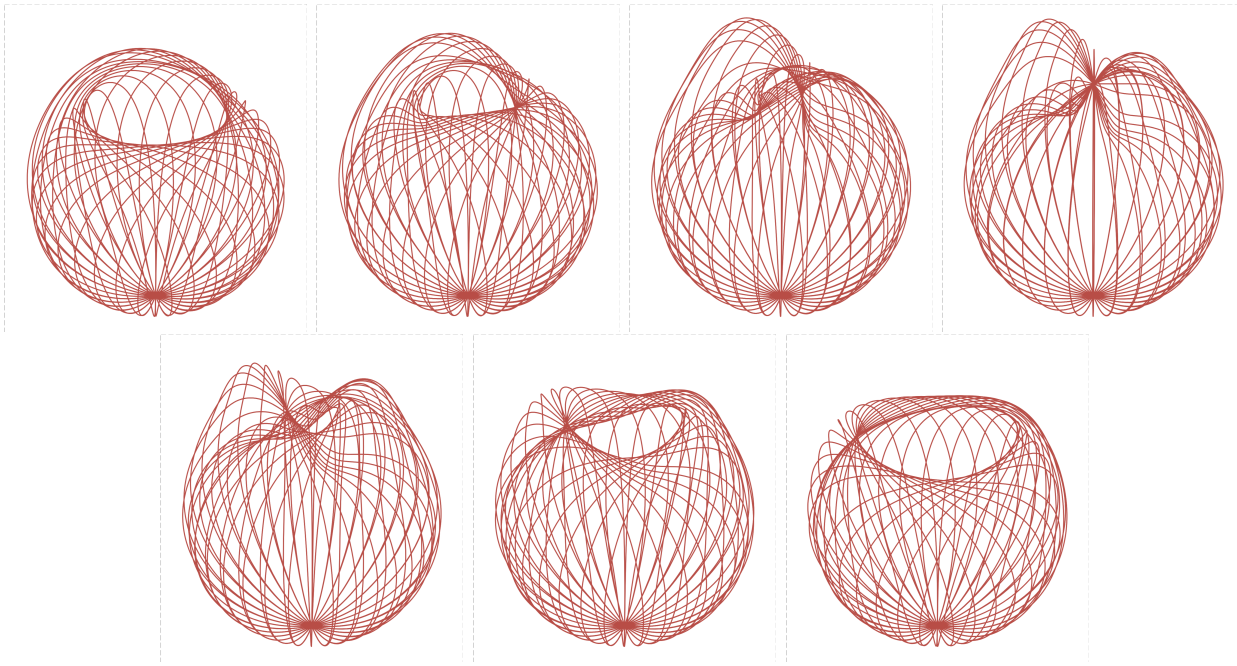

Figure 6. The Bednorz-Bednorz sphere eversion (for $n = 2$) modeled in Desmos with 3D contour lines. The Desmos graph is available at `https://www.desmos.com/calculator/2ru2ctxesf`.

is parameterized by its height and angle $(h, \phi) \in \mathbb{R} \times [0, 2\pi]$.

$$x_1 = t\cos(\phi) + p\sin((n-1)\phi) - h\sin(\phi)$$
$$y_1 = t\sin(\phi) + p\cos((n-1)\phi) + h\cos(\phi)$$
$$z_1 = h\sin(n\phi) - \frac{t}{n}\cos(n\phi) - qth$$

Here, p and q are arbitrary values with $q \geq 0$, and t is the "timestamp" of the homotopy. We use $p = 0$ and $q = \frac{1}{2}$ in Figure 5.

Now, similar to how a sphere can be thought of as a cylinder with the upper and lower bases smoothly "capped off," we can extend this eversion to the sphere by "capping it off." First, we define the following functions, which parametrize an intermediate "wormhole" that we need before properly defining the sphere.

$$x_2 = x_1(\xi + \eta(x^2 + y^2))^{-\kappa}$$
$$y_2 = y_1(\xi + \eta(x^2 + y^2))^{-\kappa}$$
$$z_2 = z_1(\xi + \eta(x^2 + y^2))^{-1}$$

Here, x_1, y_1, and z_1 are the equations from the cylindrical eversion, ξ and η are arbitrary values with $\xi, \eta \geq 0$, and $\kappa = \frac{n-1}{2n}$.

Now, we can finally parameterize the actual Bednorz-Bednorz sphere eversion.

$$x_3 = \frac{x_2 e^{\gamma z_2}}{\alpha + \beta(x_2^2 + y + y_2^2)}$$
$$y_3 = \frac{y_2 e^{\gamma z_2}}{\alpha + \beta(x_2^2 + y_2^2)}$$
$$z_3 = \frac{\alpha - \beta(x_2^2 + y_2^2)}{\alpha + \beta(x_2^2 + y_2^2)} \cdot \frac{e^{\gamma z_2}}{\gamma} - \frac{\alpha - \beta}{\gamma(\alpha + \beta)}$$

Here, x_2, y_2, and z_2 are the previous equations, α and β are arbitrary values with $\alpha, \beta \geq 0$, and $\gamma = 2\sqrt{\alpha\beta}$. We use $\alpha = 1$ and $\beta = \frac{1}{25}$ in Figure 6.

Remark 5.2. Note that the sphere begins and ends with one of its poles twisted and pushed towards the other. It is not difficult to see how this can be regularly homotopied back to the sphere and the everted sphere.

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