Vector Bundles, Characteristic Classes, and a Journey to Cobordism

Nicholas James and Ari Krishna

June 12, 2023

Contents

1 Introduction

When studying manifold theory, there are two approaches one can take. The first is to endow the manifold with a metric in order to study the manifold locally. This would be more geometric in scope, and is commonly referred to as differential geometry. The second, however, is when you do not endow such a metric on the manifold. Instead, one observes the *global* properties of the manifold itself in comparison to other manifolds, usually endowing some kind of smooth structure and comparing the manifolds through smooth maps. This approach, called differential topology, concerns the study of smooth manifolds and the smooth maps (diffeomorphisms) between them. While studying such a subject, it is natural to inquire about the possibility of classifying all manifolds of each given dimension up to diffeomorphism.

As fate would have it, this proves to be quite a daunting task to approach head-on. It is known that manifolds of dimension at least 4 can admit any finitely-presented group as a fundamental group, and the "group isomorphism problem" of algorithmically determining whether two finitely-presented groups are isomorphic is undecidable. Needless to say, the classification of compact manifolds is not a trivial problem, and requires more advanced machinery to solve in some substantial capacity.

But where does one begin with such a problem? Two simpler options unfold: one must either restrict to special cases of this problem, or introduce a coarser invariant than diffeomorphism. In the 1950s, René Thom made substantial breakthroughs on the latter front, earning him the Fields Medal in 1958. He came up with notions of equivalence that were both rich and easily-computable, an equivalence known today as *cobordism*. We will examine cobordism of compact manifolds in its unoriented and oriented flavors, but this journey to cobordism is quite the rocky road.

In order to reach our desired destination, some mathematical machinery will be assumed of the reader. The authors assume that the reader is familiar with algebraic topology, but if the reader would like a refresher on these topics, we point them to the work of tom Dieck in [Die08]. By using homology and cohomology theory, we will engage in an exciting exposition on characteristic classes covered in [MS74] and [Die08] and analyze how it relates back to our question of cobordism. We will also freely reference the results on cobordism contained in [Hir76] and [Wal16].

2 Vector Bundles

When one studies differential topology for the first time, they learn about the so-called "tangent bundle" and its many constructions. However, there is a far more general notion of the tangent bundle that is encountered in a larger setting, called the vector bundle over a topological space. In this section, we briefly discuss the most interesting properties that they possess. Our first treatment of vector bundles begins with constructions from [MS74], [Spi05], and [Lee13], where we will combine said treatments to create a more cohesive language to describe the vector bundle.

2.1 Vector Bundles and Basic Notions

When studying differential topology or geometry, one usually treats vector bundles as *generalizations* of a tangent bundle TM for some manifold M. However, we will do the opposite and treat the vector bundle as the protagonist of our story.

Definition 2.1.1. Let B denote a fixed topological space, called the ***base space***. A ***vector bundle*** ξ over B is a five-tuple

$$\xi = (E(\xi), \pi, B, \oplus, \odot)$$

consisting of the following objects and properties:

1. A topological space $E = E(\xi)$ called the ***total space*** of ξ.
2. A continuous map $\pi : E \to B$ called the ***projection map*** from E to B.
3. For every $x \in B$, the two maps $\oplus$ and $\odot$ are defined as

$$\oplus : \bigcup_{x\in B} \pi^{-1}(x) \times \pi^{-1}(x) \to E \ , \ \ \oplus(v, w) = v + w \in E.$$
$$\odot : \mathbb{R} \times E \to E \ , \ \ \odot(a, v) = a \cdot v = av \in E$$

   such that $\pi^{-1}(x)$ is a vector space of $\mathbb{R}$.

4. The ***local triviality condition*** is satisfied: for every point $x \in B$, there exists a neighborhood $U \subseteq B$, an integer $n \geq 0$, and a homeomorphism

$$h : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

   which is a vector space isomorphism from each $x \times \mathbb{R}^n$ to the vector space $\pi^{-1}(x)$, for all $x \in U$.

In reference to condition (4) of Definition 2.1.1, we call the pair (U, h) the ***local coordinate system for*** ξ ***about*** x. It may very well be the case that U is equivalent to the base space B, and in such an instance we say that the vector bundle ξ is a ***trivial bundle***. Furthermore, we call the vector space $\pi^{-1}(x)$ the ***fiber*** of the projection π over x. In some texts it is denoted as F_x or $F_x(\xi)$, and we make no apologies for using all three notations interchangeably.

Example. The trivial bundle ϵ_B^n consists of a total space $B \times \mathbb{R}^n$ with a projection map

$$\pi : B \times \mathbb{R}^n \to B \ , \ \pi(x, a) = x,$$

and with its vector space structures in the fibers being defined by

$$\begin{aligned} \oplus\Big(\odot (t_1, (x, a_1)), \odot(t_2, (x, a_2))\Big) &= \odot(t_1, (x, a_1)) + \odot(t_2, (x, a_2)) \\ &= (x, t_1 a_1 + t_2 a_2). \end{aligned}$$

This is well-defined since we can set $(x, t_1 a_1 + t_2 a_2) = (x, a_3) \in B \times \mathbb{R}^n$.

Definition 2.1.2. A vector bundle ξ is an $\mathbb{R}^n$***-bundle*** if $\pi^{-1}(x)$ is an n-dimensional real vector space for every $x \in B$.

The distinction between an $\mathbb{R}^n$-bundle and an arbitrary vector bundle is subtle but must be distinguished, because we must remember that the dimension of $\pi^{-1}(x)$ is a local condition. In other words, it is dependent on each element $x \in B$. We impose another definition in a similar way.

Definition 2.1.3. A vector bundle ξ is ***smooth*** if its total space E and base space B are smooth manifolds, its projection map π is smooth, and for every $x \in B$ the local coordinate system (U, h) about x has h as a diffeomorphism.

Example. The ***tangent bundle*** τ_M of a smooth manifold M is an example of a smooth vector bundle. Its total space is the disjoint union of tangent spaces

$$TM = \coprod_{x \in M} T_x M = \{(x, v) \mid x \in M, v \text{ is tangent to } M \text{ at } x\},$$

and the projection map $\pi : TM \to M$ is defined by $\pi(x, v) = x$; the vector space structure of $\pi^{-1}(x)$ is defined by

$$\begin{aligned} \oplus\Big(\odot (t_1, (x, v_1)), \odot(t_2, (x, v_2))\Big) &= \odot(t_1, (x, v_1)) + \odot(t_2, (x, v_2)) \\ &= (x, t_1 v_1 + t_2 v_2). \end{aligned}$$

Using the above definition as motivation, we now use vector bundles to generalize the notion of vector fields.

Definition 2.1.4. A ***section*** of a vector bundle ξ with base space B is a continuous function $\sigma : B \to E(\xi)$ which takes each $x \in B$ into the corresponding fiber $F_x(\xi)$. If $\sigma(x)$ is a nonzero vector of $F_x(\xi)$ for each $x \in B$, we say that σ is ***nowhere zero***. Furthermore, let $U \subseteq B$ be open. We define the ***zero section*** of ξ over U as the function φ which takes every element $x \in U$ to the zero element of $\pi^{-1}(x)$.

Example. If $E(\xi) = TM$ for some smooth manifold M, then s is a vector field.

It is important to realize how $\sigma(x)$ is a vector contained in $\pi^{-1}(x)$. In other words, we can apply ideas such as linear independence to these vectors, and we do so in the following definition.

Definition 2.1.5. The sections $(\sigma_1, \ldots, \sigma_n)$ are ***nowhere dependent*** if, for each $x \in B$, the vectors $(\sigma_1(x), \sigma_2(x), \ldots, \sigma_n(x))$ are linearly independent.

Now that we have discussed vector bundles as an object, we can now describe how vector bundles relate to other vector bundles under isomorphisms.

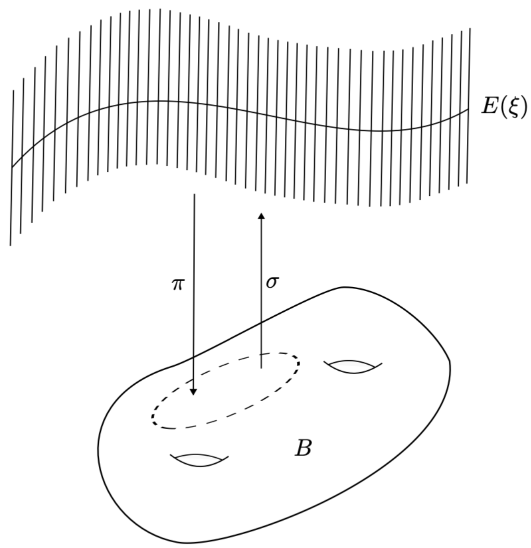


Figure 2.1. Depiction of a section σ of a vector bundle ξ.

Definition 2.1.6. Let ξ and η be vector bundles. We say that ξ and η are ***isomorphic***, which we denote by $\xi \cong \eta$, if there is a homeomorphism

$$f : E(\xi) \to E(\eta)$$

between the total spaces mapping each vector space $F_x(\xi)$ in $E(\xi)$ to its corresponding vector space $F_x(\eta)$ in $E(\eta)$.

One might initially question why f being a homeomorphism is necessary. Is it too strong of a condition to have instead of something such as, say, continuity? Well, the choice is irrelevant since both are treated the same way in this context. Here is why.

Theorem 2.1.7. *Let ξ and η be vector bundles over B and let $f : E(\xi) \to E(\eta)$ be a continuous function which maps each vector space $F_x(\xi)$ isomorphically to the corresponding vector space $F_x(\eta)$. Then f is a homeomorphism and $\xi \cong \eta$.*

Proof. Fix $x \in B$ and choose local coordinate systems (U, g) for ξ and (V, h) for η with $x \in U \cap V$. We show that the map

$$h^{-1} \circ f \circ g : (U \cap V) \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n$$

is a homeomorphism. For the forward direction, define the map by $h^{-1}(f(g(x, a))) = (x, b)$. Then we see that each component of $b = \{b_1, \ldots, b_n\}$ can be expressed as a linear combination such that

$$b_i = \sum_j f_{ij}(x) a_j$$

where $[f_{ij}(x)]$ denotes a nonsingiular matrix of real numbers. The entries of this matrix depend continuously on x, so $h^{-1} \circ f \circ g$ is continuous. To show that $(h^{-1} \circ f \circ g)^{-1}$ is continuous, let $[F_{ji}(x)]$ denote the inverse matrix of $[f_{ij}(x)]$. Then we see that

$$(h^{-1} \circ f \circ g)^{-1}(x, b) = (g^{-1} \circ f^{-1} \circ h)(x, b) = (x, a)$$

where each real number $a = (a_1, \ldots, a_n)$ can be written as the linear combination

$$a_j = \sum_i F_{ji}(x) b_i.$$

Since the entries $F_{ji}(x)$ depend continuously on the entries $f_{ij}(x)$ which depend continuously on x, it follows that the entries $F_{ji}(x)$ depend continuously on b. This implies that $g^{-1} \circ f^{-1} \circ h$ is continuous, and the proof is complete. ■

Thus, our notion of bundle isomorphisms is precise. But what about restricting bundles, and making bundles out of other bundles? This is what we shall look at next.

Definition 2.1.8. Let ξ be a vector bundle with projection $\pi : E \to B$ and let $\overline{B} \subseteq B$. By setting $\overline{E} = \pi^{-1}(\overline{B})$ and letting $\overline{\pi} : \overline{E} \to \overline{B}$ be the restriction of π to $\overline{E}$, one obtains the ***restriction*** of ξ to its ***subbase space*** $\overline{B}$ which we denote as the $\xi|_{\overline{B}}$. We write this formally as the five-tuple

$$\begin{aligned} \xi|_{\overline{B}} &= \left(\pi^{-1}(\overline{B}), \pi|\pi^{-1}(\overline{B}), \bigcup_{x \in \overline{B}} \pi^{-1}(x) \times \pi^{-1}(x), \odot|\mathbb{R} \times \pi^{-1}(\overline{B}) \right) \\ &= \left(\overline{E}, \pi|\overline{E}, \bigcup_{x \in \overline{B}} \pi^{-1}(x) \times \pi^{-1}(x), \odot|\mathbb{R} \times \overline{E} \right). \end{aligned}$$

It is important to note that the local triviality condition of $\xi|_{\overline{B}}$ is satisfied since it really is just a local condition in the first place. Thus, one can safely view $\xi|_{\overline{B}}$ as a vector bundle in its own right.

Definition 2.1.9. Let ξ be a vector bundle over B and let B' be an arbitrary topological space. Given any map $f : B' \to B$, one can construct what is called the ***pullback (or induced) vector bundle*** $f^*\xi$ over B' in the following way.

1. The total space E' of $f^*\xi$ is a subset of $B' \times E$, consisting of all pairs (x', e) such that $f(x') = \pi(e)$.

2. The projective map $\pi' : E' \to B'$ is defined by $\pi'(x', e) = x'$, and the following diagram commutes where $\hat{f}(x', e) = e$:

$$\begin{array}{ccc} E' & \xrightarrow{\hat{f}} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

3. The vector space structure on $\pi'(x')$ is defined by

$$\begin{aligned} \oplus\Big(\odot(t_1, (x', e_1)), \odot(t_2, (x', e_2)) \Big) &= \odot(t_1, (x', e_1)) + \odot(t_2, (x', e_2)) \\ &= (x', t_1 e_1 + t_2 e_2). \end{aligned}$$

4. The local triviality condition is satisfied as well. If (U, h) is a local coordinate system for ξ, then set $U' = f^{-1}(U)$ and define

$$h' : U' \times \mathbb{R}^n \to \pi'^{-1}(U'),$$

by $h'(x', a) = (x', h(f(x'), a))$. By construction we have that (U', h') is a local coordinate system for $f^*\xi$ and $f^*\xi$ is locally trivial.

The construction of the pullback bundle presents two remarks. Firstly, it is immediate that $f^*\xi$ is smooth if ξ is smooth with f being a smooth map. Second, the map $\hat{f}$ presents a more general construction of a vector bundle isomorphism.

Definition 2.1.10. Let ξ and η be vector bundles over base spaces B and B' respectively. A ***bundle map*** from ξ to η is a continuous function

$$f : E(\xi) \to E(\eta)$$

which maps each vector space $F_x(\xi)$ isomorphically to one of the vector spaces $F_{x'}(\eta)$.

It is important to note the subtle difference between a vector bundle isomorphism and a bundle map. Bundle maps are different than isomorphisms because the base space is *not* fixed. Furthermore, by setting $\overline{f}(x) = x'$ it is clear from the above definition that

$$\overline{f} : B(\xi) \to B(\eta)$$

is continuous as well. Using this we prove the following.

Theorem 2.1.11. *If $f : E(\xi) \to E(\eta)$ is a bundle map, and if $\overline{f} : B(\xi) \to B(\eta)$ is the corresponding base space map, then $\xi \cong \overline{f}^* \eta$.*

Proof. Consider the function $h : E(\xi) \to E(\overline{f}^* \eta)$ defined by $h(e) = (\pi(e), f(e))$ where π denotes the projection of ξ. Because h is continuous and maps each vector space $F_x(\xi)$ isomorphically to its corresponding vector space $F_x(\eta)$, it follows from Theorem 2.1.7 that h is a homeomorphism, which completes the proof. ■

2.2 New Vector Bundles From Old

Given that the vector bundle is the central object of this paper, we must also elaborate on the various new kinds of bundles we can construct from old ones. One such example is immediate.

Definition 2.2.1. Let $\mathscr{F} = \{\xi_i\}_{i \in \Lambda}$ be a family of vector bundles for some index set Λ, and let $\pi_i : E(\xi_i) \to B(\xi_i)$ be the projection maps for each vector bundle $\xi_i \in \mathscr{F}$. Then the ***Cartesian product bundle*** $\prod_{i \in \Lambda} \xi_i$ is defined to be the vector bundle with projection map

$$\prod_{i \in \Lambda} \pi_i : \prod_{i \in \Lambda} E(\xi_i) \to \prod_{i \in \Lambda} B(\xi_i)$$

where each fiber (or vector space)

$$\left(\prod_{i \in \Lambda} \pi_i \right)^{-1} (x_1, \ldots, x_i) = \prod_{i \in \Lambda} F_x(\xi_i),$$

is given the obvious vector space structure, and it is easy to see that $\prod_{i \in \Lambda} \xi_i$ is locally trivial because each $\xi_i \in \mathscr{F}$ is.

Numerous properties are still preserved when we take the product of vector bundles. For example, if M is the Cartesian product of smooth manifolds M_i, then the tangent bundle τ_M is isomorphic to the cartesian product of each tangent bundle τ_{M_i}.

Definition 2.2.2. Let ξ_1 and ξ_2 be vector bundles over the same base space B, and define the map

$$\Delta : B \to B \times B$$

to be the ***diagonal embedding*** of ξ_1 and ξ_2. Then the pullback bundle with respect to Δ

$$\Delta^*(\xi_1 \times \xi_2) = \xi_1 \oplus \xi_2$$

is called the ***Whitney sum*** of ξ_1 and ξ_2. Each fiber $F_x(\xi_1 \oplus \xi_2)$ is cononically isomorphic to the direct sum $F_x(\xi_1) \oplus F_x(\xi_2)$.

Whitney sums allow us to build new vector bundles out of old ones in a different way from, say, Cartesian products or pullbacks. To see what the Whitney sum does in particular, we need to look at vector bundles as objects once again.

Definition 2.2.3. Let η and ξ be vector bundles over the same base space B with $E(\xi) \subset E(\eta)$. We say that ξ is a ***subbundle*** of η if each fiber $F_x(\xi)$ is a subspace of the corresponding fiber $F_x(\eta)$. We denote this as $\xi \subset \eta$.

If $\xi_1, \xi_2 \subset \eta$ such that $F_x(\eta) = F_x(\xi_1) \oplus F_x(\xi_2)$ then $\eta \cong \xi_1 \oplus \xi_2$. To see this, let $f : E(\xi_1 \oplus \xi_2) \to E(\eta)$ by $f(x, e_1, e_2) = e_1 + e_2$. This is continuous, and by Theorem 2.1.7 the result follows. But this raises another question. If $\xi \subset \eta$, is there another subbundle $\rho \subset \eta$ such that $\eta = \xi \oplus \rho$? The answer is yes, and to see why we need another definition.

Definition 2.2.4. Let $\xi \subset \eta$ be a subbundle of η over B, and let $F_x(\xi^{\perp})$ denote the subspace of $F_x(\eta)$ consisting of all vectors v such that $v \cdot w = 0$ for all $w \in F_x(\xi)$. In this case, we call $\xi^{\perp}$ the ***orthogonal complement*** of ξ with respect to η. Furthermore, we write $E(\xi^{\perp})$ as the union of all of the $F_x(\xi^{\perp})$.

We use the above to answer our question from before. It can be proved (see [MS74]) that $E(\xi^{\perp})$ is the total space of the subbundle $\xi^{\perp}$ and $\eta \cong \xi \oplus \xi^{\perp}$. Thus, we can always insure that η can be "broken down" into a Whitney sum of subbundles. We present one last definition which will be very important throughout this exposition.

Definition 2.2.5. Let M and N be smooth manifolds and let N be endowed with a ***Riemannian metric*** $\mu : TN \to \mathbb{R}$; we call (N, μ) a ***Riemannian manifold*** in this case. Then the tangent bundle τ_M is a subbundle of $\tau_{N|M}$, and its orthogonal compliment $\nu = \tau_M^{\perp}$ is called the ***normal bundle*** of M with respect to N.

One of the most important results that we will use with normal bundles is the following theorem, known as the Tubular Neighborhood Theorem. We state it without proof, but we guide the reader to a proof that can be found in [Hir76].

Theorem 2.2.6 (Tubular Neighborhood Theorem)**.** *There exists an open neighborhood of U in M which is diffeomorphic to the total space of the normal bundle under a diffeomorphism which maps each point $x \in U$ to the zero normal vector at x.*

Another important thing to note is the intimate relationship that Whitney sums and normal bundles have with each other.

Theorem 2.2.7. *For any smooth submanifold M of a smooth Riemannian manifold N, the normal bundle ν is defined and $\tau_M \oplus \nu \cong \tau_{N|M}$.*

Proof. This follows immediately from the fact that $\nu = \tau_M^{\perp}$. ■

While we have stated the above definition with a Riemannian metric, we define a new kind of metric in order to define our next type of vector bundle.

Definition 2.2.8. A ***Euclidean vector bundle*** is a real vector bundle ξ together with a continuous function $\mu : E(\xi) \to \mathbb{R}$ such the restriction of μ to each fiber in $E(\xi)$ is positive definite and ***quadratic***. In other words, we can write

$$\mu(v) = \sum_i \ell_i(v)\ell_i'(v)$$

where ℓ and ℓ' are linear functions.

Another important example is when we define a vector bundle with respect to a quotient space. We do this using projective spaces.

Definition 2.2.9. Let $E(\gamma_n^1)$ be a subset of $\mathbb{P}^n \times \mathbb{R}^{n+1}$ consisting of pairs $(\pm x, v\}$ such that each vector v is a multiple of x. Define the projection $\pi : E(\gamma_n^1) \to \mathbb{P}^n$ by $\{\pm x, v\} \mapsto \{\pm x\}$. Then each fiber $\pi^{-1}(\{\pm x\})$ is a line through x and $-x$ in $\mathbb{R}^{n+1}$ and is given the usual vector space structure. The vector bundle γ_n^1 is called the ***cononical line bundle*** over its base space $\mathbb{P}^n$.

While the above material has certainly been notable, we have reached a dead-end with the current technology we have at the moment. In order to progress any further, we must turn to the language of category theory. By doing so, we can define more operations on vector bundles, as well as new kinds of vector bundles.

Definition 2.2.10. Let $\mathbf{Vect}_{\mathbb{R}}$ be the category of all finite dimensional vector spaces V over $\mathbb{R}$, and all of the isomorphisms between them. We define a covariant functor $\mathcal{T} : \mathbf{Vect}_{\mathbb{R}} \times \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}$ as an operation which assigns

- To each pair of vector spaces $V, W \in \mathbf{Vect}_{\mathbb{R}}$ a vector space $\mathcal{T}(V, W) \in \mathbf{Vect}_{\mathbb{R}}$.
- To each pair of isomorphisms $f : V \to V'$ and $g : W \to W'$ an isomorphism

$$\mathcal{T}(f, g) : \mathcal{T}(V, W) \to \mathcal{T}(V', W')$$

  such that $\mathcal{T}(\mathrm{id}_V, \mathrm{id}_W) = \mathrm{id}_{\mathcal{T}(V,W)}$ and the composition of isomorphisms behaves in such a way that

$$\mathcal{T}(f_1 \circ f_2, g_1 \circ g_2) = \mathcal{T}(f_1, g_1) \circ \mathcal{T}(f_2, g_2).$$

This functor is said to be ***continuous*** if $\mathcal{T}(f, g)$ depends continuously on f and g.

This seems quite unrelated though, so we must make this relevant in the context of vector bundles. We do this by recalling that $F_x(\xi)$ is a vector space.

Definition 2.2.11. Let $\mathcal{T} : \mathbf{Vect}_{\mathbb{R}} \times \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{R}}$ be a continuous functor of n variables, and let $\xi_1, \dots, \xi_n$ be a collection of vector bundles over the same base space B. Then for each $x \in B$, let

$$F_x = \mathcal{T}(F_x(\xi_1), F_x(\xi_2), \dots, F_x(\xi_n)).$$

Let the total space E be equivalent to the disjoint union of the spaces such that

$$E = \coprod_{x \in B} F_x,$$

and define the projection map $\pi : E \to B$ by $\pi(F_x) = x$. Using the necessary cononical topology for E, this forms a tangent bundle and we denote it by $\mathcal{T}(\xi_1, \dots, \xi_n)$.

Our reward for this category-theoretic work is that we can now define the ***tensor product*** $\xi \otimes \eta$ of vector bundles ξ and η by using the tensor product functor. Furthermore, if we use the duality functor $V \mapsto \mathrm{Hom}(V, \mathbb{R})$, we obtain the functor $\xi \mapsto \mathrm{Hom}(\xi, \epsilon^1)$ which is called the ***dual vector bundle*** of ξ. There are many other extensions of this idea depending on one's choice of functor, and we leave it to the reader to explore.

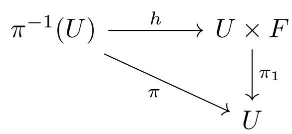
2.3 A Word on Fiber Bundles

Although vector bundles are quite a generalization themselves, we can take this one step further by defining the fiber bundle.

Definition 2.3.1. Let B and F be fixed topological spaces. A ***fiber bundle over*** B ***with model fiber*** F is a four-tuple

$$\phi = (E, B, \pi, F)$$

with a surjective continuous projection $\pi : E \to B$ satisfying the following property: for every $x \in B$, there exists a neighborhood U of x and a homeomorphism $h : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:



Remark 2.3.2. We can endow a smooth structure on ϕ if we so desire. If all four components of ϕ are smooth, and h is a diffeomorphism, then we say that ϕ is a ***smooth fiber bundle***.

A vector bundle is a special case of a fiber bundle, in the sense that we invoke a vector space structure on the fibers $\pi^{-1}(x)$ and F. By removing the vector space structure for fiber bundles, we are able to generalize vector bundles to a much larger class of spaces.

Definition 2.3.3. A ***trivial fiber bundle*** ϕ is one that admits a local trivialization over the entire base space

$$h_{\text{global}} : \pi^{-1}(B) \to B \times F$$

which we call a ***global trivialization***. We say that ϕ is ***smoothly trivial*** if ϕ is a smooth fiber bundle and the global trivialization h is a diffeomorphism.

While vector bundles are a classic example of fiber bundles, there are other simpler examples to consider as well.

Example. Every product space $B \times F$ is a fiber bundle (called a ***product fiber bundle***) with projection $\pi_1 : B \times F \to B$. It has a global trivialization $h_{\text{global}} : B \times F \to B \times F$, so every product fiber bundle is trivial.

Example. Every covering space is a fiber bundle, where the projection is a local homeomorphism. Its model fiber is a discrete space.

3 Stiefel-Whitney Classes

The idea of characteristic classes is to create topological invariants on a vector bundle by associating it to some kind of cohomology class. Regarding the notation used in this section, we direct the reader to the appendices of [MS74].

3.1 Definitions and Immediate Results

Definition 3.1.1. Let ξ be a vector bundle. The ***Stiefel-Whitney classes*** of ξ are a sequence of cohomology classes

$$w_i(\xi) \in \mathrm{H}^i(B(\xi); \mathbb{Z}/2\mathbb{Z})$$

where the following axioms are satisfied:

1. The class $w_0(\xi)$ is equal to the unit element $1 \in \mathrm{H}^0(B(\xi); \mathbb{Z}/2\mathbb{Z})$.
2. If ξ is an $\mathbb{R}^n$-bundle and $i > n$, then $w_i(\xi) = 0$.
3. If $f : B(\xi) \to B(\eta)$ is covered by a bundle map from ξ to η, then $w_i(\xi) = f^* w_i(\eta)$.
4. If ξ and η are vector bundles over B, then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \smile w_{k-i}(\eta) = \sum_{i+j=k} w_i(\xi) w_j(\eta).$$

5. If γ_1^1 is the line bundle over the circle $\mathbb{P}^1$, the first Stiefel-Whitney class $w_1(\gamma_1^1) \neq 0$.

There are a couple of things that need to be clarified regarding the statement of Definition 3.1.1. Firstly, why the choice of coefficients in $\mathbb{Z}/2\mathbb{Z}$? Well, recall the following result from algebraic topology.

Theorem 3.1.2 ((Poincaré Duality). *Let M be a compact and orientable n-dimensional manifold. Then $\mathrm{H}^i(M; \Lambda) \cong \mathrm{H}_{n-i}(M; \Lambda)$ by the mapping $a \mapsto a \cap \mu_M$.*

Thus, the orientability of $B(\xi)$ takes center stage. If it is orientable, there is nothing to worry about and we can move freely between homology and cohomology to establish invariants. But what happens if $B(\xi)$ is not orientable?

Theorem 3.1.3. *If M is an n-dimensional unorientable manifold, then $\mathrm{H}_n(M; \mathbb{Z}) = 0$.*

In other words, something goes really wrong when we look at unorientable manifolds with cohomology class coefficients in $\mathbb{Z}$. But this does not happen in $\mathbb{Z}/2\mathbb{Z}$. In fact, it is the only possible choice of coefficients where Theorem 3.1.2 still holds for unorientable manifolds.

Remark 3.1.4. An intuitive way of looking at the final Stiefel-Whitney classe axiom is that the nonzero cohomology "detects" the twist in the Möbius bundle; that is, the ***obstruction*** to it assuming the global structure of a product space.

Now that we have defined the axioms we need (and while it is not obvious that such a class can be constructed), it is time to produce some results. There is one that is quite immediate.

Theorem 3.1.5. *If $\xi \cong \eta$ then $w_i(\xi) = w_i(\eta)$.*

Proof. The isomorphism $\xi \cong \eta$ induces the identity map as its base map, which means that by axiom (3),

$$w_i(\xi) = f^* w_i(\eta) = \mathrm{id}^* w_i(\eta) = w_i(\eta).$$

This completes the proof. ■

This establishes Stiefel-Whitney classes as an invariant of vector bundles. What is interesting about Stiefel-Whitney classes, however, is that they encode more topological invariants beyond this. We continue our exploration to establish these invariants.

Theorem 3.1.6. *Let ϵ_B^n be a trivial vector bundle over B. Then $w_i(\epsilon_B^n) = 0$ for $i > 0$.*

Proof. By definition we know that the total space of ϵ_B^n is equivalent to $B \times \mathbb{R}^n$ with projection map $\pi_1 : B \times \mathbb{R}^n \to B$. Consider another projection map $\pi_2 : B \times \mathbb{R}^n \to \mathbb{R}^n$.

$$\begin{array}{ccc} B \times \mathbb{R}^n & \xrightarrow{\pi_2} & \mathbb{R}^n \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_3} \\ B & \xrightarrow{\pi_4} & \{*\} \end{array}$$

From the above diagram, we use two additional projections from B and $\mathbb{R}^n$ to the singleton $\{a\}$, or rather that there is a bundle map to a vector bundle over a point. However, the higher cohomology groups of a point are trivial, and by taking the pullback we see that $w_i(\epsilon_B^n) = 0$ for $i > 0$ by axiom (3). ■

Another result that is important to note is when the Whitney sum of two Stiefel-Whitney classes cancels out, which allows for another vanishing criterion.

Theorem 3.1.7. *Let ϵ_B^n be a trivial vector bundle. Then for any vector bundle ξ over some base space, the Whitney sum $w_i(\epsilon_B^n \oplus \xi) = w_i(\xi)$.*

Proof. Applying axiom (4), we see that

$$w_i(\epsilon_B^n \oplus \xi) = \sum_{j=0}^{i} w_j(\epsilon_B^n) \smile w_{i-j}(\xi) = w_i(\xi)$$

since $w_j(\epsilon_B^n)$ vanishes for $j > 0$ by Theorem 3.1. This completes the proof. ■

3.2 Stiefel-Whitney Numbers and Unoriented Cobordism

Because of the invariant properties that Stiefel-Whitney classes possess, how are we supposed to compare them? This is where a new type of invariant comes into play, and we focus more on its theoretical applications rather than its uses in computation. To begin, we define a notational definition.

Definition 3.2.1. Let M be a closed smooth n-dimensional manifold. Recall (see [MS74]) that there is a unique fundamental homology class $\mu_M \in \mathrm{H}_n(M;\mathbb{Z}/2\mathbb{Z})$ using mod 2 coefficients. For any cohomology class $v \in \mathrm{H}^n(M;\mathbb{Z}/2\mathbb{Z})$ we define the ***Kronecker index*** $v[M] = \langle v, \mu_M \rangle \in \mathbb{Z}/2\mathbb{Z}$ as the composition of the cap product

$$\cap : \mathrm{H}^i(M;\mathbb{Z}/2\mathbb{Z}) \otimes \mathrm{H}_n(M;\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}_{n-i}(M;\mathbb{Z}/2\mathbb{Z}) \ , \ v \cap \mu_M \in \mathrm{H}_0(M;\mathbb{Z}/2\mathbb{Z})$$

with the point evaluation map $p : \mathrm{H}_{n-i}(X;\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$.

The idea of the Kronecker index is that if we take an element from a cohomology group and another from its corresponding homology group, we can operate on those two elements to get an element contained in their coefficient domain. The next definition has a surprising usage of the tangent bundle.

Definition 3.2.2. Let $r_1, \ldots, r_n$ be non-negative integers such that $\sum_{i=1}^{n} ir_i = n$. If $\xi = \tau_M$ for some manifold M with the previously described conditions, then we define the ***Stiefel-Whitney number of*** M as the Kronecker index of the monomial $w_1(\tau_M)^{r_1} \ldots w_n(\tau_M)^{r_n}$. In other words, it is equivalent to

$$w_1(\tau_M)^{r_1} \cdots w_n(\tau_M)^{r_n}[M] = \langle w_1(\tau_M)^{r_1} \cdots w_n(\tau_M)^{r_n}, \mu_M \rangle.$$

As previously stated, we are not necessarily concerned with the computation of these numbers. Rather, we are interested in how these numbers produce results related to cobordism. A result from Pontryagin is a good start.

Theorem 3.2.3 (Pontryagin)**.** *If M' is a smooth compact $(n+1)$-manifold with boundary equal to M (which has the previous conditions imposed on it), then the Stiefel-Whitney numbers of M are all zero.*

Proof. Denote the fundamental homology class of the pair (M', M) as $\mu_{M'} \in \mathrm{H}_{n+1}(M', M;\mathbb{Z}/2\mathbb{Z})$. Then the homomorphism

$$\partial : \mathrm{H}_{n+1}(M', M;\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}_n(M;\mathbb{Z}/2\mathbb{Z})$$

maps $\mu_{M'}$ to the fundamental homology class μ_M. By taking any arbitrary cohomology class $v \in \mathrm{H}^n(M;\mathbb{Z}/2\mathbb{Z})$, we see that the identity

$$\langle v, \partial\mu_{M'} \rangle = \langle \delta v, \mu_{M'} \rangle \qquad (\star)$$

holds where $\delta : \mathrm{H}^n(M;\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^{n+1}(M', M;\mathbb{Z}/2\mathbb{Z})$ (see the appendix of [MS74]). Consider the tangent bundle $\tau_{M'|M}$, and choose a Euclidean metric on $\tau_{M'}$. Then there is a unique outward normal vector along M which spans the trivial line bundle $\epsilon^1_{\mathbb{P}^n}$, and by Theorem 2.2.7 we have $\tau_{M'|M} \cong \tau_M \oplus \epsilon^1_{\mathbb{P}^n}$. By Theorem 3.1 we know that $w_i(\tau_M \oplus \epsilon^1_{\mathbb{P}^n}) \cong w_i(\tau_M)$, so by axiom (3) we see that $w_i(\tau_M) \cong w_i(\tau_{M'|M})$. Because the sequence

$$\mathrm{H}^n(M';\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\subset} \mathrm{H}^n(M;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\delta} \mathrm{H}^{n+1}(M', M;\mathbb{Z}/2\mathbb{Z})$$

is exact, we see that $\delta(w_1^{r_1} \ldots w_n^{r_n}) = 0$, and by $(\star)$ this implies that

$$\langle (w_1^{r_1} \cdots w_n^{r_n}), \partial\mu_{M'} \rangle = \langle \delta(w_1^{r_1} \cdots w_n^{r_n}), \mu_{M'} \rangle$$

so all Stiefel-Whitney numbers of M are zero. ∎

The converse is also true, actually, although we will not prove this here. This surprisingly nontrivial result is due to Thom, and it is stated as follows.

Theorem 3.2.4 (Thom)**.** *If all Stiefel-Whitney numbers of M (assuming the above conditions) are zero, then M is the boundary of some smooth compact manifold.*

The two theorems from Pontryagin and Thom allow us to establish our first result on cobordism. To do this, we first define what we mean by "cobordism".

Theorem 3.2.5. *Two smooth n-manifolds M and M' belong to the same* ***unoriented cobordism class*** *if and only if their disjoint union $M \sqcup M'$ is the boundary of a smooth compact $(n+1)$-manifold.*

Cobordism is extremely easy to visualize, which is why this theory is so rich. In essence, two smooth n-manifolds are *cobordant* if their (disjoint, so there is no overlap) union consists of all the boundary points of an $(n+1)$-dimensional manifold. The entire rest of this paper is devoted to developing enough machinery to determine when two manifolds M and M' satisfy this simple condition. For our first result, we state the following.

Definition 3.2.6. Two smooth (closed) n-manifolds M and M' belong to the same unoriented cobordism class if and only if all of their Stiefel-Whitney numbers are equal.

Proof. This is immediate from Theorems 3.2.3 and 3.2.4. ■

Our journey to cobordism has just begun from this point onward. Later, we will see how Pontryagin classes solve the oriented case, and how we can use homotopy theory to solve the cobordism problem once and for all.

4 Universal Bundles and Oriented Bundles

4.1 Universal Bundles

One of the most common vector bundles that we will use involves slicing Euclidean space with n-dimensional planes. We briefly formalize this idea with a special kind of base space, which we define now. Consider the set of all n-dimensional planes through the origin of $\mathbb{R}^{n+k}$. This is a topological space, and we topologize it in the following way.

Definition 4.1.1. An ***n-frame*** in $\mathbb{R}^{n+k}$ is an n-tuple of linearly independent vectors in $\mathbb{R}^{n+k}$. The collection of all n-frames in $\mathbb{R}^{n+k}$ forms an open subset of the cartesian product

$$\underbrace{\mathbb{R}^{n+k} \times \ldots \times \mathbb{R}^{n+k}}_{n\text{-many times}}$$

which we call the ***Stiefel manifold*** $V_n(\mathbb{R}^{n+k})$.

Using this idea of n-frames, we can construct a (canonical) function ρ from $V_n(\mathbb{R}^{n+k})$ to our desired set by mapping each n-frame to each n-space that it spans. In particular, we define strong continuity by saying that a subset U in this space is open if and only if its preimage $\rho^{-1}(U)$ is open in $V_n(\mathbb{R}^{n+k})$. With its topology defined, we state our space's definition in full.

Definition 4.1.2. The space $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ is the nk-dimensional compact manifold consisting of all n-dimensional planes through the origin of $\mathbb{R}^{n+k}$, called the ***Grassmann manifold***.

The justifications that are needed in order to show that $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ is an nk-dimensional and compact manifold are contained in [MS74], but we omit the details. Using the idea of the Grassmann manifold, our next objective is to talk about the vector bundle which we take over $\mathrm{Gr}_n(\mathbb{R}^{n+k})$.

Definition 4.1.3. The ***universal bundle*** $\gamma^n(\mathbb{R}^{n+k})$ over $\mathrm{Gr}_n(\mathbb{R}^{n+k})$ possesses a total space

$$E(\gamma^n(\mathbb{R}^{n+k})) = (n\text{-plane in } \mathbb{R}^{n+k}, \text{vector in that } n\text{-plane}) \subseteq \mathrm{Gr}_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}.$$

It has the projection map $\pi : E \to \mathrm{Gr}_n(\mathbb{R}^{n+k})$ defined by $\pi(X, x) = X$, with its fibers being defined by

$$\begin{aligned} \oplus\Big(\odot (t_1, (X, x_1)), \odot(t_2, (X, x_2))\Big) &= \odot(t_1, (X, x_1)) + \odot(t_2, (X, x_2)) \\ &= (X, t_1x_1 + t_2x_2). \end{aligned}$$

Constructed in this way, we see that local triviality is satisfied as well. For the details, consider the exposition contained in [MS74].

The "universal" name for this bundle seems quite strong, but it is certainly warranted. The reason why is because every $\mathbb{R}^n$-bundle over a compact base space can be mapped into $\gamma^n(\mathbb{R}^{n+k})$ as long as k is sufficiently large. We will not prove this here, but we encourage the reader to see the proof in [MS74].

Definition 4.1.4. The ***infinite Grassmann manifold*** $\mathrm{Gr}_n = \mathrm{Gr}_n(\mathbb{R}^\infty)$ is the set of all n-dimensional subspaces of $\mathbb{R}^\infty$, whose topology is obtained from the the direct limit of the sequence

$$\mathrm{Gr}_n(\mathbb{R}^n) \subset \mathrm{Gr}_n(\mathbb{R}^{n+1}) \subset \mathrm{Gr}_n(\mathbb{R}^{n+2}) \subset \mathrm{Gr}_n(\mathbb{R}^{n+3}) \cdots .$$

Hence, a subset $\mathrm{Gr}_n(\mathbb{R}^i) \subseteq \mathrm{Gr}_n$ is open or closed if and only if its itersesection $\mathrm{Gr}_n(\mathbb{R}^i) \cap \mathrm{Gr}_n(\mathbb{R}^j)$ is open or closed for some other $\mathrm{Gr}_n(\mathbb{R}^j) \subseteq \mathrm{Gr}_n$. It is evident that

$$\mathrm{Gr}_n = \bigcup_{i=0}^{\infty} \mathrm{Gr}_n(\mathbb{R}^{n+i})$$

such that the universal bundle γ^n is the vector bundle with base space Gr_n

4.2 Oriented Bundles and Euler Classes

Until now, we have only looked at cohomology with coefficients in $\mathbb{Z}/2\mathbb{Z}$. While we have gone quite far with this choice of coefficients (especially Theorem 3.2.6), we will be able to incorporate even more information if we allow our coefficients to be in $\mathbb{Z}$. It turns out that in order to do this, we will need to impose the additional structure of an *orientation* on our vector bundles. To get things started, let us discuss the notion of an orientation on a single finite-dimensional vector space V.

Definition 4.2.1. An ***orientation*** of a real vector space V of dimension $n > 0$ is an equivalence class of bases, where two (ordered) bases $v_1, \ldots, v_n$ and $v_1', ..., v_n'$ are said to be equivalent if and only if the matrix $[a_{ij}]$ defined by the equation $v_i' = \sum a_{ij} v_j$ has positive determinant. It follows that every such vector space has exactly two possible orientations.

Just like how manifolds are orientable, we can also define something similar for vector bundles. Put simply, in order to make a vector bundle ξ orientable, we make each fiber orientable as a vector space.

Definition 4.2.2 (Vector Space Version)**.** Let ξ be a vector bundle of fiber dimension $n > 0$. An ***orientation*** for ξ is a function which assigns to each fiber F of ξ an orientation as a vector space, subject to the following compatibility condition: for every $x \in B$, there exists a local coordinate chart (M, θ) containing x with

$$\theta : M \times \mathbb{R}^n \to \pi^{-1}(M)$$

such that for each fiber $\pi^{-1}(x) = F$ over M, the homomorphism $y \mapsto \theta(x, y)$ from $\mathbb{R}^n$ to F is orientation preserving.

In order to do anything involving characteristic classes with our new definition above, we have to rephrase Definition 4.2.2 in the (equivalent) language of cohomology theory. In algebraic topology, we can orient an arbitrary n-simplex Σ^n in such a way that we can label its vertices and form an orientation with them.

Definition 4.2.3. Let Σ^n be an n-simplex that is linearly embedded into an n-dimensional vector space V, composed of ordered vertices $A_0, \ldots, A_n$. Each vector in V connecting A_i to A_{i+1} for $i \in \{0, \ldots, n-1\}$ forms a basis vector $\mathcal{B}_i$ for V. The set of ordered bases $\{\mathcal{B}\}_i$ comprise the ***corresponding orientation*** of the vector space V.

An important thing to note here is that a choice of orientation for V corresponds to a choice of the two possible generators of the homology group $\mathrm{H}_n(V, V_0; \mathbb{Z})$. With such a choice, we have to pick one, and we define rigorously what we mean by "preference" when making such a choice.

Definition 4.2.4. Let Δ^n be a standard n-simplex, let V be an oriented n-dimensional vector space, and let $\sigma : \Delta^n \to V$ be an orientation preserving linear embedding which maps the barycenter of Δ^n to V_0. Then σ is a singular n-simplex in $Z_n(V, V_0; \mathbb{Z})$, and the homology class of σ is the ***preferred generator*** μ_V for $\mathrm{H}_n(V, V_0; \mathbb{Z})$. A similar construction unfolds for the cohomology group $\mathrm{H}^n(V, V_0; \mathbb{Z})$ and we denote its preferred generator as κ_V where it satisfies the identity $\langle \kappa_V, \mu_V \rangle = 1$.

Now we are prepared to translate Definition 4.2.2 into the language of cohomology. In essence, what we will do is assign each fiber a preferred generator in the cohomology class of the fiber and its boundary.

Definition 4.2.5 (Cohomology Version)**.** Let ξ be a vector bundle of fiber dimension $n > 0$. An ***orientation*** for ξ is a function which assigns to each fiber F of ξ a preferred generator $\kappa_F \in \mathrm{H}^n(F, F_0; \mathbb{Z})$ subject to the following local compatibility condition: for every $x \in B$ there exists a neighborhood U of x and a cohomology class $\kappa \in \mathrm{H}^n(\pi^{-1}(U), \pi^{-1}(U)_0; \mathbb{Z})$ such that, for every fiber F over U, the restriction $\kappa|_{F,F_0} \in \mathrm{H}^n(F, F_0; \mathbb{Z})$ is equivalent to κ_F.

But this is only for fibers of the vector bundle. Does this definition still hold when we consider the entire total space E of ξ? The answer is yes, and we will state this result without proof.

Theorem 4.2.6 (Thom)**.** *Let ξ be an oriented $\mathbb{R}^n$ bundle. Then the cohomology group* $\mathrm{H}^n(E(\xi), E_0; \mathbb{Z}) = 0$ *for $i < n$, and* $\mathrm{H}^n(E(\xi), E_0; \mathbb{Z})$ *contains precisely one cohomology class κ (called the **Thom class**) whose restriction $\kappa|_{F,F_0} \in \mathrm{H}^n(F, F_0; \mathbb{Z})$ is equal to the preferred generator κ_F for every fiber F of ξ. Additionally,*

$$\mathrm{H}^k(E(\xi); \mathbb{Z}) \cong \mathrm{H}^{k+n}(E(\xi), E_0; \mathbb{Z})$$

for each integer k by the isomorphism $\rho \mapsto \rho \smile \kappa$.

Let B be the base space of ξ. Then the above implies that $\mathrm{H}^k(B(\xi); \mathbb{Z}) \cong \mathrm{H}^{k+n}(E(\xi), E_0; \mathbb{Z})$, and in a more general sense we call this the ***Thom isomorphism***

$$\Theta : \mathrm{H}^k(B(\xi); \mathbb{Z}) \to \mathrm{H}^{k+n}(E(\xi), E_0; \mathbb{Z}) \ , \ \Theta(\rho) = \pi^*\rho \smile \kappa$$

where π^* is the canonical isomorphism $\pi^* : \mathrm{H}^k(B(\xi); \mathbb{Z}) \to \mathrm{H}^k(E(\xi); \mathbb{Z})$. Surprisingly, all of this theory is actually used to create a new kind of characteristic class. Let ξ be an oriented $\mathbb{R}^n$-bundle with total and base spaces E and B respectively. By considering the cohomology class $\kappa \in \mathrm{H}^n(E(\xi), E_0; \mathbb{Z})$, we can restrict this to the total space to achieve a new cohomology class $\kappa|_E \in \mathrm{H}^n(E(\xi); \mathbb{Z}) \cong \mathrm{H}^n(B(\xi); \mathbb{Z})$.

Definition 4.2.7. Let ξ be an oriented $\mathbb{R}^n$-bundle. The ***Euler class*** of ξ is the cohomology class $\mathrm{e}(\xi) \in \mathrm{H}^n(B(\xi); \mathbb{Z})$ which corresponds to $\kappa|_E$ with respect to the canonical isomorphism π^* written above.

For lack of a better phrase, it should soon become apparent to the reader that Euler classes should behave "as they should be" so to speak. We see why this is in the following results.

Theorem 4.2.8 (Naturality and Sign Change)**.** *If $f : B \to B'$ is covered by an orientation preserving bundle map $\xi \to \xi'$, then $\mathrm{e}(\xi) = f^*\mathrm{e}(\xi')$. Additionally, if the orientation of ξ is reversed, then $\mathrm{e}(\xi)$ changes sign.*

Proof. These are immediate. The first follows from the uniqueness of the Thom class in Theorem 4.2.6, and the second follows from the "parity" property of the fundamental class (and thus the Thom class). ∎

Another key result is when the fibers of ξ have odd dimension. When this happens, the Euler class of ξ changes sign.

Theorem 4.2.9. *If ξ is an oriented $\mathbb{R}^n$-bundle with odd fiber dimension, then* $\mathrm{e}(\xi) = -\mathrm{e}(\xi)$*.*

Proof. Because we defined the Thom isomorphism as

$$\Theta : \mathrm{H}^n(B(\xi);\mathbb{Z}) \to \mathrm{H}^{n+k}(E(\xi), E_0;\mathbb{Z}),$$

we can take $\mathrm{e}(\xi) \in \mathrm{H}^n(B(\xi);\mathbb{Z})$ such that

$$\Theta(\mathrm{e}(\xi)) = \pi^*\mathrm{e}(\xi) \smile \kappa = \kappa|_E \smile \kappa = \kappa \smile \kappa$$

and taking inverses gives $\mathrm{e}(\xi) = \Theta^{-1}(\kappa \smile \kappa)$. But note the identity which states

$$\kappa \smile \kappa = (-1)^{(\dim \kappa)(\dim \kappa)}\kappa \smile \kappa,$$

so we see that $\mathrm{e}(\xi)$ has order 2 when ξ has odd fiber dimension. ∎

There is one last result that we will cover, since it basically concludes our discussion of the "arithmetic" properties of the Euler class.

Theorem 4.2.10. *If ξ and ξ' are oriented $\mathbb{R}^n$-bundles, then the Euler class of the product bundle* $\mathrm{e}(\xi \times \xi') = \mathrm{e}(\xi) \times \mathrm{e}(\xi')$ *and the Euler class of the Whitney sum is* $\mathrm{e}(\xi \oplus \xi') = \mathrm{e}(\xi) \smile e(\xi')$*.*

Proof. Let ξ and ξ' be of fiber dimension m and n respectively. Notice that (by accounting for sign) the fundamental cohomology class of the product is equivalent to

$$\kappa(\xi \times \xi') = (-1)^{mn}\kappa(\xi) \times \kappa(\xi'). \qquad (\star)$$

Hence, by applying the restriction homomorphism

$$\mathrm{H}^{m+n}(E(\xi) \times E'(\xi), E_0 \times E_0';\mathbb{Z}) \longrightarrow \mathrm{H}^{m+n}(E(\xi), E'(\xi);\mathbb{Z}) \cong \mathrm{H}^{m+n}(B(\xi), B'(\xi);\mathbb{Z})$$

to both sides of $(\star)$, we get the desired relation $\mathrm{e}(\xi \times \xi') = (-1)^{mn}\mathrm{e}(\xi) \times \mathrm{e}(\xi')$. For the second, suppose that ξ and ξ' are oriented $\mathbb{R}^n$-bundles over the same base space B. Then "lifting" the previous product back to B via the diagonal embedding $B \to B \times B$ gives $\mathrm{e}(\xi \oplus \xi') = \mathrm{e}(\xi) \smile \mathrm{e}(\xi')$. ∎

We will be using Euler classes and their properties to define other kinds of characteristic classes, mainly Chern and Pontryagin classes.

4.3 Gysin Sequences

Since orientation is the topic at hand, we will study an important sequence of cohomology classes of the base space and total space of an oriented $\mathbb{R}^n$-bundle ξ.

Definition 4.3.1. Let ξ be an oriented $\mathbb{R}^n$-bundle. The ***Gysin sequence*** of ξ is the exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(B(\xi);\mathbb{Z}) \xrightarrow{\smile \mathrm{e}} \mathrm{H}^{n+i}(B(\xi);\mathbb{Z}) \xrightarrow{\pi_0^*} \mathrm{H}^{n+i}(E_0;\mathbb{Z}) \longrightarrow \mathrm{H}^{i+1}(B(\xi);\mathbb{Z}) \xrightarrow{\smile \mathrm{e}} \cdots$$

where $\smile e$ denotes the map $\rho \mapsto \rho \smile \mathrm{e}(\xi)$.

Hopefully the importance of the Gysin sequence is clear. If such a Gysin sequence exists, we can effortlessly relate the cohomology of the base space of ξ to its total space, a connection that will prove vital when building more complicated characteristic classes. Thankfully, this question of existence is not an issue at all.

Theorem 4.3.2. *Every oriented $\mathbb{R}^n$-bundle ξ has a Gysin sequence.*

Proof. We start with the following exact sequence of cohomology groups

$$\cdots \longrightarrow \mathrm{H}^j(E(\xi), E_0; \mathbb{Z}) \longrightarrow \mathrm{H}^j(E(\xi); \mathbb{Z}) \longrightarrow \mathrm{H}^j(E_0; \mathbb{Z}) \xrightarrow{\delta} \mathrm{H}^{j+1}(E(\xi), E_0; \mathbb{Z}) \longrightarrow \cdots .$$

By Theorem 4.2.6 we have $\mathrm{H}^{j-n}(E(\xi); \mathbb{Z}) \cong \mathrm{H}^j(E(\xi), E_0; \mathbb{Z})$, so we can replace these components of this sequence to get another exact sequence

$$\cdots \longrightarrow \mathrm{H}^{j-n}(E(\xi); \mathbb{Z}) \xrightarrow{\ell} \mathrm{H}^j(E(\xi); \mathbb{Z}) \longrightarrow \mathrm{H}^j(E_0; \mathbb{Z}) \longrightarrow \mathrm{H}^{j-n+1}(E(\xi); \mathbb{Z}) \longrightarrow \cdots$$

where $\ell(\rho) = \rho \smile \kappa|_E$. Because $\kappa|_E \in \mathrm{H}^j(E(\xi); \mathbb{Z})$ corresponds to the Euler class $\mathrm{e}(\xi) \in \mathrm{H}^j(B(\xi); \mathbb{Z})$, we finish by seeing that this results in the desired exact sequence

$$\cdots \longrightarrow \mathrm{H}^i(B(\xi); \mathbb{Z}) \xrightarrow{\smile e} \mathrm{H}^{n+i}(B(\xi); \mathbb{Z}) \xrightarrow{\pi_0^*} \mathrm{H}^{n+i}(E_0; \mathbb{Z}) \longrightarrow \mathrm{H}^{i+1}(B(\xi); \mathbb{Z}) \xrightarrow{\smile e} \cdots$$

by substituting $\mathrm{H}^j(B(\xi); \mathbb{Z})$ in place of $\mathrm{H}^j(E(\xi); \mathbb{Z})$ and setting $i = j - n$. ■

5 Complex Vector Bundles and Chern Classes

In this chapter, we move onward in the rabbit hole of characteristic classes and into the realm of complex vector bundles. Our goal is to associate new kinds of cohomology classes with complex vector bundles, which we call Chern classes. These complex characteristic classes serve as the final stepping stone to Pontryagin classes.

5.1 Complex Vector Bundles

We have worked with vector bundles whose fibers have the structure of finite-dimensional vector spaces over $\mathbb{R}$. Now we extend naturally to vector bundles over $\mathbb{C}$.

Definition 5.1.1. A ***complex n-vector bundle*** ω over B (or equivalently a $\mathbb{C}^n$-bundle) consists of a topological space E and a projection map $\pi : E \to B$, together with the structure of a complex vector space in each fiber $\pi^{-1}(x)$, subject to the following local triviality condition: for every point $x \in B$, there exists a neighborhood $U \subseteq B$, an integer $n \geq 0$, and a homeomorphism

$$h : U \times \mathbb{C}^n \to \pi^{-1}(U)$$

which is a vector space isomorphism from each $x \times \mathbb{C}^n$ to the vector space $\pi^{-1}(x)$, for all $x \in U$.

This definition should not be shocking, considering how similar it is to its real analogue. The main difference with complex vector bundles is how we construct them. One way to do this is to start with a $\mathbb{R}^{2n}$-bundle, and attempt to give each fiber the additional structure of a complex vector space.

Definition 5.1.2. Let ξ be a $\mathbb{R}^{2n}$-bundle. A ***complex structure*** on ξ is a continuous mapping

$$\mathfrak{J} : E(\xi) \to E(\xi)$$

satisfying the identity $(\mathfrak{J} \circ \mathfrak{J})(v) = -v$ for every $v \in E(\xi)$.

Using Definition 5.1.2, we transform each $F_x(\xi)$ into a complex vector space by setting

$$zv = (x + iy)v = xv + \mathfrak{J}(yv)$$

for every $z \in \mathbb{C}$, and since local triviality follows naturally we see that our $\mathbb{R}^{2n}$-bundle ξ is also a complex vector bundle. But what happens in the opposite case? Can we take some $\mathbb{C}^n$-bundle ω and turn it into a $\mathbb{R}^{2n}$-bundle? Yes.

Definition 5.1.3. Let ω be a $\mathbb{C}^n$-bundle. By omitting $\mathfrak{J}$, we see that each fiber $F_x(\omega)$ is a $2n$-dimensional real vector space in its own right, forming the ***underlying real $\mathbb{R}^{2n}$-bundle*** $\omega_{\mathbb{R}}$.

Now that we have established what complex vector bundles are, we can now examine cohomology classes associated with them.

5.2 Chern Classes

In short, Chern classes are characteristic classes that are associated with complex vector bundles. Before we define them, we must first remark on the orientability of complex vector bundles, since we still have not dropped the orientability condition.

Theorem 5.2.1. *If ω is a complex vector bundle, then the underlying real vector bundle $\omega_{\mathbb{R}}$ has a canonical preferred orientation.*

Proof. Let V be any finite dimensional complex vector space. Choosing an ordered basis $v_1, \ldots, v_n$ for V over $\mathbb{C}$, note that the vectors $v_1, iv_1, v_2, iv_2, \ldots, v_n, iv_n$ form a new $\mathbb{R}$-basis for the underlying vector space $V_{\mathbb{R}}$. This basis determines the required orientation for $V_{\mathbb{R}}$, and it is important to realize that this is not basis-dependent; since $\mathrm{GL}_n(\mathbb{C})$ is connected, we can pass from any given complex basis to any other complex basis by some continuous deformation, which leaves fixed the induced orientation. This completes the proof. ■

Remark 5.2.2. If we apply this lemma to tangent bundles of manifolds, it follows that any complex manifold has a canonical preferred orientation. Every orientation for the tangent bundle of a manifold gives rise to a unique orientation on the manifold itself. Now, if ω is a complex vector bundle, applying this construction to every fiber of ω yields the required orientation for $\omega_{\mathbb{R}}$.

As an application, for any $\mathbb{C}^n$-bundle ω over a base space B, we see that the Euler class $\mathrm{e}(\omega_{\mathbb{R}}) \in \mathrm{H}^{2n}(B(\omega); \mathbb{Z})$ is well-defined. This allows us to conclude the following theorem.

Theorem 5.2.3. *If ω' is a $\mathbb{C}^m$-bundle over the same base space B, then*

$$\mathrm{e}\left((\omega \oplus \omega')_{\mathbb{R}}\right) = \mathrm{e}\left(\omega_{\mathbb{R}}\right) \mathrm{e}\left(\omega'_{\mathbb{R}}\right).$$

Proof. This is just a simple argument among bases. If $a_1, \ldots, a_n$ is a basis of a fiber F for ω, and $b_1, \ldots, b_m$ is a basis of the corresponding fiber F' of ω', then the preferred orientation $a_1, ia_1, \ldots, a_n, ia_n$ for $F_{\mathbb{R}}$ followed by the preferred orientation $b_1, ib_1, \ldots, b_m, ib_m$ for $F'_{\mathbb{R}}$ yields the preferred orientation $a_1, ia_1, \ldots, ia_n, b_1, ib_1, \ldots, ib_m$ for $(F \oplus F')_{\mathbb{R}}$. Thus $\omega_{\mathbb{R}} \oplus \omega'_{\mathbb{R}}$ is isomorphic as an oriented bundle to $(\omega \oplus \omega')_{\mathbb{R}}$. ■

While Euclidean metrics play an important role in the study of real vector bundles, there is a suitable analogue for complex vector bundles.

Definition 5.2.4. A ***Hermitian metric*** on a complex vector bundle ω is a Euclidean metric

$$v \mapsto |v|^2 \geq 0$$

on the underlying real vector bundle, which also satisfies the identity

$$|iv| = |v|.$$

Given a Hermitian metric, there exists a unique complex-valued inner product defined by the identity

$$\langle v, w \rangle = \frac{1}{2}\left(|v+w|^2 - |v|^2 - |w|^2\right) + \frac{1}{2} i \left(|v+iw|^2 - |v|^2 - |iw|^2\right)$$

defined for v and w in the same fiber of ω, which

1. is complex linear as a function of v for fixed w,

2. is conjugate linear as a function of w for fixed v (that is $\langle v, \lambda w \rangle = \bar{\lambda} \langle v, w \rangle$, and

3. satisfies $\langle v, v\rangle = |v|^2$. The two vectors v and w are said to be orthogonal if this complex inner product $\langle v, w\rangle$ is zero. The Hermitian identity

$$\langle w, v\rangle = \overline{\langle v, w\rangle}$$

is easily verified, hence v is orthogonal to w if and only if w is orthogonal to v.

Remark 5.2.5. Note that if B is paracompact, then every complex vector bundle over B admits a Hermitian metric.

Here is an inductive approach to defining characteristic classes for a $\mathbb{C}^n$-bundle ω. It is first necessary to construct a canonical $\mathbb{C}^{n-1}$-bundle ω_0 over the deleted total space E_0. (Note that $E_0 = E_0(\omega)$ denotes the set of all non-zero vectors in the total space $E(\omega) = E(\omega_{\mathbb{R}})$.) A point in E_0 is specified by a fiber F of ω together with a non-zero vector v in that fiber. First, suppose that a Hermitian metric has been specified on ω. Then the fiber of ω_0 over v is by definition the orthogonal complement of v in the vector space F. This is a complex vector space of dimension $n-1$, and these vector spaces can be considered as the fibers of a new vector bundle ω_0 over E_0.

Remark 5.2.6. We do not necessarily need the Hermitian metric here. The fiber of ω_0 over v can be defined as the quotient vector space $F/\mathbb{C}v$ where $\mathbb{C}v$ is the 1-dimensional subspace spanned by the vector $v \neq 0$. In the presence of a Hermitian metric, it is of course clear that this quotient space is canonically isomorphic to the orthogonal complement of v in F.

Second, note that by Theorem 4.3.2 an oriented $\mathbb{R}^{2n}$-bundle ξ possesses an exact Gysin sequence of the form

$$\dots \longrightarrow \mathrm{H}^{i-2n}(B(\xi);\mathbb{Z}) \xrightarrow{\smile e} \mathrm{H}^{i}(B(\xi);\mathbb{Z}) \xrightarrow{\pi_0^*} \mathrm{H}^{i}(E_0;\mathbb{Z}) \longrightarrow \mathrm{H}^{i-2n+1}(B(\xi);\mathbb{Z}) \longrightarrow \dots.$$

For $i < 2n-1$, the cohomology groups $\mathrm{H}^{i-2n}(B(\xi);\mathbb{Z})$ and $\mathrm{H}^{i-2n+1}(B(\xi);\mathbb{Z})$ are zero. This implies that $\pi_0^* : \mathrm{H}^{i}(B(\xi);\mathbb{Z}) \longrightarrow \mathrm{H}^{i}(E_0;\mathbb{Z})$ is an isomorphism.

Definition 5.2.7. The ***Chern classes*** $\mathrm{c}_i(\omega) \in \mathrm{H}^{2i}(B(\omega);\mathbb{Z})$ are defined as follows, by induction on the complex dimension n of ω. The top Chern class $\mathrm{c}_n(\omega)$ is equal to the Euler class $\mathrm{e}(\omega_{\mathbb{R}})$. For all $i < n$, we set the Chern class equal to

$$\mathrm{c}_i(\omega) = \pi_0^{*-1}\mathrm{c}_i(\omega_0).$$

Finally, for $i > n$ the class $\mathrm{c}_i(\omega)$ is defined to be zero.

Remark 5.2.8. Note that the definition of the Chern class makes sense for $i < n$ since $\pi_0^* : \mathrm{H}^{2i}(B(\xi);\mathbb{Z}) \longrightarrow \mathrm{H}^{2i}(E_0;\mathbb{Z})$ is an isomorphism for $i < n$.

The formal sum $\mathrm{c}(\omega) = 1 + \mathrm{c}_1(\omega) + \dots + \mathrm{c}_n(\omega)$ in the ring $\mathrm{H}^{\Pi}(B;\mathbb{Z})$ is called the ***total Chern class*** of ω. We see that $\mathrm{c}(\omega)$ is a unit, so it has an inverse

$$\mathrm{c}(\omega)^{-1} = 1 - \mathrm{c}_1(\omega) + \left(\mathrm{c}_1(\omega)^2 - \mathrm{c}_2(\omega)\right) + \dots$$

which is well-defined. Our next goal is to use the definition above to build some new theory.

Theorem 5.2.9 (Naturality)**.** *If $f : B \longrightarrow B'$ is covered by a bundle map from the $\mathbb{C}^n$-plane bundle ω over B to the $\mathbb{C}^n$-bundle ω' over B', then* $\mathrm{c}(\omega) = f^*\mathrm{c}(\omega')$*.*

Proof. We proceed with a proof by induction on n. The top Chern class is natural, $\mathrm{c}_n(\omega) = f^*\mathrm{c}_n(\omega')$, since Euler classes are natural by 4.2.8. To prove the corresponding statement for lower Chern classes, first note that the bundle map $\omega \longrightarrow \omega'$ gives rise to a map

$$f_0 : E_0(\omega) \longrightarrow E_0(\omega')$$

which is covered by a bundle map $\omega_0 \longrightarrow \omega_0'$ of $\mathbb{C}^{n-1}$-bundles. Hence $c_i(\omega_0) = f_0^* c_i(\omega_0')$ by hypothesis. Using the commutative diagram below

$$\begin{array}{ccc} E_0(\omega) & \xrightarrow{f_0} & E_0(\omega') \\ \downarrow{\scriptstyle \pi_0'} & & \downarrow{\scriptstyle \pi_0} \\ B & \xrightarrow{f} & B' \end{array}$$

and the identities $c_i(\omega_0) = \pi_0^* c_i(\omega)$ and $c_i(\omega_0') = \pi_0'^* c_i(\omega')$ where π_0' is an isomorphism for $i < n$, it follows that $\mathrm{c}_i(\omega) = f^* \mathrm{c}_i(\omega')$. ■

Theorem 5.2.10. *If ϵ_B^k is the trivial $\mathbb{C}^k$-bundle over $B = B(\omega)$, then* $\mathrm{c}\left(\omega \oplus \epsilon_B^k\right) = \mathrm{c}(\omega)$

Proof. We prove this only for the base case of $k = 1$, since this generalizes easily to all other k by induction. Let $\phi = \omega \oplus \epsilon_B^1$. Since the $\mathbb{C}^{n+1}$-bundle ϕ has a non-zero cross-section, it follows that the top Chern class $\mathrm{c}_{n+1}(\phi) = \mathrm{e}(\phi_{\mathbb{R}})$ is zero, and hence equal to $\mathrm{c}_{n+1}(\omega)$. Let $s : B \longrightarrow E_0\left(\omega \oplus \epsilon_B^1\right)$ be the natural cross-section. Clearly s is covered by a bundle map $\omega \longrightarrow \phi_0$, hence

$$s^* \mathrm{c}_i(\phi_0) = \mathrm{c}_i(\omega)$$

by the preceding lemma. Substituting $\pi_0^* \mathrm{c}_i(\phi)$ for $\mathrm{c}_i(\phi_0)$, and using the formula $s^* \circ \pi_0^* = \mathrm{id}$, it follows that $\mathrm{c}_i(\phi) = \mathrm{c}_i(\omega)$, as desired. ■

5.3 Complex Grassmann Manifolds

Like most results in this chapter, we will now observe how Grassmann manifolds have complex analogues, too. Like the real case, we define the ***complex Grassmann manifold*** $\mathrm{Gr}_n\left(\mathbb{C}^{n+k}\right)$ to be the set of all complex n-planes through the origin in $\mathbb{C}^{n+k}$. The complex Grassmann $\mathrm{Gr}_n\left(\mathbb{C}^{n+k}\right)$ also has a natural complex structure, which we can view as a complex analytic manifold of (complex) dimension nk. Furthermore there is a canonical $\mathbb{C}^n$-bundle which we denote by $\gamma^n = \gamma^n\left(\mathbb{C}^{n+k}\right)$ over $\mathrm{Gr}_n\left(\mathbb{C}^{n+k}\right)$. By definition, the total space of γ^n consists of all pairs (X, v) where X is a complex n-plane through the origin in $\mathbb{C}^{n+k}$ and v is a vector in X.

Applying the Gysin sequence to the canonical line bundle $\gamma^1 = \gamma^1\left(\mathbb{C}^{k+1}\right)$ over $\mathbb{P}^k(\mathbb{C})$, and using the fact that $\mathrm{c}_1\left(\gamma^1\right) = \mathrm{e}\left(\gamma_{\mathbb{R}}^1\right)$, we have

$$\ldots \longrightarrow \mathrm{H}^{i+1}(E_0; \mathbb{Z}) \longrightarrow \mathrm{H}^i\left(\mathbb{P}^k(\mathbb{C}); \mathbb{Z}\right) \xrightarrow{\mathrm{c}_1} \mathrm{H}^{i+2}\left(\mathbb{P}^k(\mathbb{C}); \mathbb{Z}\right) \xrightarrow{\pi_0^*} \mathrm{H}^{i+2}(E_0; \mathbb{Z}) \longrightarrow \ldots.$$

Furthermore, we can describe the space $E_0 = E_0\left(\gamma^1\left(\mathbb{C}^{k+1}\right)\right)$ as the set of all pairs

(line through 0 in $\mathbb{C}^{k+1}$, some nonzero vector in that line)

This can be identified with $\mathbb{C}^{k+1} \backslash \{0\}$, and hence has the same homotopy type as the unit sphere $\mathbb{S}^{2k+1}$. Due to this, our Gysin sequence reduces to

$$0 \longrightarrow \mathrm{H}^i\left(\mathbb{P}^k(\mathbb{C}); \mathbb{Z}\right) \xrightarrow{\smile_1} \mathrm{H}^{i+2}\left(\mathbb{P}^k(\mathbb{C}); \mathbb{Z}\right) \longrightarrow 0$$

for $0 \leq i \leq 2k - 2$, which implies that

$$\mathrm{H}^0\left(\mathbb{P}^k(\mathbb{C}); \mathbb{Z}\right) \cong \mathrm{H}^2\left(\mathbb{P}^k(\mathbb{C}); \mathbb{Z}\right) \cong \ldots \cong \mathrm{H}^{2k}\left(\mathbb{P}^k(\mathbb{C}); \mathbb{Z}\right).$$

Since $\mathbb{P}^k(\mathbb{C})$ is clearly connected, it follows that each $\mathrm{H}^{2i}\left(\mathbb{P}^k(\mathbb{C})\right)$ is infinite cyclic generated $\mathrm{c}_1\left(\gamma^1\right)^i$ for $i \leq k$. Similarly

$$\mathrm{H}^1\left(\mathbb{P}^k(\mathbb{C});\mathbb{Z}\right) \cong \mathrm{H}^3\left(\mathbb{P}^k(\mathbb{C});\mathbb{Z}\right) \cong \ldots \cong \mathrm{H}^{2k-1}\left(\mathbb{P}^k(\mathbb{C});\mathbb{Z}\right)$$

and using the portion of the Gysin sequence,

$$\cdots \longrightarrow \mathrm{H}^{-1}\left(\mathbb{P}^k(\mathbb{C});\mathbb{Z}\right) \longrightarrow \mathrm{H}^1\left(\mathbb{P}^k(\mathbb{C});\mathbb{Z}\right) \longrightarrow \mathrm{H}^1\left(E_0;\mathbb{Z}\right) \longrightarrow \ldots$$

we see that these odd-dimensional groups are all zero. In other words, we have obtained the following theorem.

Theorem 5.3.1. *The cohomology ring* $\mathrm{H}^*\left(\mathbb{P}^k(\mathbb{C});\mathbb{Z}\right)$ *is a truncated polynomial ring terminating in dimension* $2k$*, and generated by the Chern class* $c_1\left(\gamma^1\left(\mathbb{C}^{k+1}\right)\right)$.

Now, let $k \to \infty$. The canonical $\mathbb{R}^n$-bundle $\gamma^n\left(\mathbb{C}^\infty\right)$ over $\mathrm{Gr}_n\left(\mathbb{C}^\infty\right)$, for sanity's sake, will be denoted as γ^n. Note that $\mathrm{H}^*\left(\mathrm{Gr}_1\left(\mathbb{C}^\infty\right)\right)$ is the polynomial ring generated by $c_1\left(\gamma^1\right)$, which allows one to state the following (for a proof, see [MS74]).

Theorem 5.3.2. *The cohomology ring* $\mathrm{H}^*\left(\mathrm{Gr}_n\left(\mathbb{C}^\infty\right);\mathbb{Z}\right)$ *is the polynomial ring over* $\mathbb{Z}$ *generated by the Chern classes* $\mathrm{c}_1\left(\gamma^n\right), \ldots, \mathrm{c}_n\left(\gamma^n\right)$*. There are no polynomial relations between these* n *generators.*

As exhibited previously for $\mathbb{R}^n$ bundles, we can prove the following result for $\mathbb{C}^n$-bundles.

Theorem 5.3.3. *Every* $\mathbb{C}^n$*-bundle over a paracompact base space* B *possesses a bundle map into the canonical* $\mathbb{C}^n$*-bundle* $\gamma^n = \gamma^n\left(\mathbb{C}^\infty\right)$ *over* $\mathrm{Gr}_n = \mathrm{Gr}_n\left(\mathbb{C}^\infty\right)$

Here is another way of saying this: every $\mathbb{C}^n$-bundle over a paracompact base B is isomorphic to an induced bundle $f^*\left(\gamma^n\right)$ for some $f : B \longrightarrow \mathrm{Gr}_n$. Just as in the real case, one can actually prove the sharper statement that two induced bundles $f^*\left(\gamma^n\right)$ and $g^*\left(\gamma^n\right)$ are isomorphic if and only if f is homotopic to g. For this reason the bundle $\gamma^n = \gamma^n\left(\mathbb{C}^\infty\right)$ is called the ***universal*** $\mathbb{C}^n$***-bundle***, and its base space $\mathrm{Gr}_n\left(\mathbb{C}^\infty\right)$ is called the ***classifying space for*** $\mathbb{C}^n$***-bundles***.

Theorem 5.3.4 (Chern Product Theorem)**.** *Consider two complex vector bundles* ω *and* ϕ *over a common paracompact base space* B*. Then the chern class of the whitney sum of* ω *and* ϕ *is multiplicative. That is,*

$$\mathrm{c}(\omega \oplus \phi) = \mathrm{c}(\omega)\mathrm{c}(\phi)$$

which expresses the total Chern class of a Whitney sum $\omega \oplus \phi$ *in terms of the total Chern classes of the individual bundles* ω *and* ϕ*.*

The proof of this theorem is surprisingly involved, and we guide the reader to a proof found in [MS74]. We remark that so-called "product formulae" are extremely common in characteristic classes, and while they are simple in statement that does not imply the simplicity of the proof!

5.4 Chern Numbers and Partitions

Just like Stiefel Whitney numbers, Chern classes have their own characteristic numbers. However, in order to properly define them, we need to formalize our understanding of partitions.

Definition 5.4.1. A ***partition*** of a non-negative integer k is an (unordered) sequence $I = i_1, \ldots, i_r$ of positive integers which sum to k. We denote the number of partitions of k as $p(k)$, which we call the ***partition function***.

Remark 5.4.2. A simple argument can be used to prove that the set of all partitions of all non-negative integers is a free commutative monoid on the generators $1, 2, 3, \ldots$.

Now, it is perfectly reasonable to wonder why partitions come into the conversation here. Recall the statement of Theorem 5.3.2, which implies that $\mathrm{H}^{2n}(\mathrm{Gr}_n(\mathbb{C}^\infty); \mathbb{Z})$ is a free abelian group of rank $p(n)$, and the product of Chern classes

$$\mathrm{c}_I(\gamma^n) = \prod_{i_a \in I} \mathrm{c}_{i_a}(\gamma^n)$$

forms a basis (or a generating set) for this cohomology group. If we take any n-dimensional compact complex manifold M, its tangent bundle τ_M can be classified by the map

$$f : M \to \mathrm{Gr}_n(\mathbb{C}^\infty)$$

such that we can take the pullback to obtain $f^*(\gamma^n) = \tau_M$. Using our new map f, we find that the fundamental homology class $\mu_{2n} \in H_{2n}(M; \mathbb{Z})$ of M induces a new homology class $f_*(\mu_{2n}) \in \mathrm{H}_{2n}(\mathrm{Gr}_n(\mathbb{C}^\infty); \mathbb{Z})$. Our next objective is to "identify" this homology class, and to do this we directly compute the $p(n)$-many Kronecker indices it possesses. These are of the form

$$\langle \mathrm{c}_{i_1}(\gamma^n) \cdots \mathrm{c}_{i_r}(\gamma^n), f_*(\mu_{2n}) \rangle.$$

As fate would have it, these Kronecker indices have an equivalent form.

Definition 5.4.3. Let $I = i_1, \ldots, i_r$ be a partition of a non-negative integer n, and let M be a compact complex n-dimensional manifold. We define the ***I-th Chern number*** as the Kronecker index

$$\begin{aligned} \mathrm{c}_I[M] &= \mathrm{c}_{i_1}(\tau_M) \cdots \mathrm{c}_{i_r}(\tau_M)[M] \\ &= \langle \mathrm{c}_{i_1}(\tau_M) \cdots \mathrm{c}_{i_r}(\tau_M), \mu_{2n} \rangle \\ &= \langle f^*(\mathrm{c}_{i_1}(\gamma^n) \cdots \mathrm{c}_{i_r}(\gamma^n)), \mu_{2n} \rangle \\ &= \langle \mathrm{c}_{i_1}(\gamma^n) \cdots \mathrm{c}_{i_r}(\gamma^n), f_*(\mu_{2n}) \rangle. \end{aligned}$$

If I is a partition of some other non-negative integer $k \neq n$, then we adopt the convention that $\mathrm{c}_I[M] = 0$.

Now that we have defined Chern numbers, our next objective is to find an algebraic way to manipulate linear combinations of them. The motivation for doing so will be made clear as we progress, but we first begin with the following (familiar) definition.

Definition 5.4.4. Let $t_1, \ldots, t_n$ be indeterminates (or variables, both are equivalent terms). A polynomial $f(t_1, \ldots, t_n)$ with integral coefficients is called ***symmetric*** if permuting the indeterminants has no effect on the value of $f(t_1, \ldots, t_n)$.

When studying differential forms, symmetric polynomials play a critical role when describing symmetric tensors. However, we will take a different direction with this mathematical technology, and instead look at the subring $\mathcal{SP} \subset \mathbb{Z}[t_1, \ldots, t_n]$ of symmetric polynomials. In order to analyze this subring, we need to establish the "building blocks" of symmetric polynomials.

Definition 5.4.5. Let $t_1, \ldots, t_n$ be indeterminates. The ***i-th elementary symmetric polynomial*** is defined as

$$\sigma_i(t_1, \ldots, t_n) = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq n} t_{j_1} t_{j_2} \cdots t_{j_i}.$$

Alternatively, if $I = i_1, \ldots, i_r$ is a partition of n, then we define it as the product

$$\sigma_I(t_1, \ldots, t_n) = \prod_{i_j \in I} \sigma_{i_j}(t_1, \ldots, t_n).$$

It turns out that every symmetric polynomial can be written as a polynomial in elementary symmetric polynomials. To formalize what we mean by this, consider the classic theorem from algebra.

Theorem 5.4.6 (Fundamental Theorem of Symmetric Polynomials)**.** *Let Λ be a commutative ring and let $f(t_1, \ldots, t_n) \in \Lambda[t_1, \ldots, t_n]$ be a symmetric polynomial. Then there exists another polynomial $g \in \Lambda[t_1, \ldots, t_n]$ such that $f(t_1, \ldots, t_n) = g(\sigma_1(t_1, \ldots, t_n), \ldots, \sigma_n(t_1, \ldots, t_n))$.*

Proof. This follows from a simple induction argument. ■

Using the Fundamental Theorem of Symmetric Polynomials, we are able to state an important corollary that justifies our previous remarks.

Corollary 5.4.7. *The subring $\mathcal{SP} \cong \mathbb{Z}[\sigma_1(t_1, \ldots, t_n), \ldots, \sigma_n(t_1, \ldots, t_n)]$.*

Now that we have an explicit isomorphism for $\mathcal{SP}$, our next task is to endow a graded ring structure on $\mathcal{SP}$. To do this, we make $\mathbb{Z}[t_1, \ldots, t_n]$ into a graded ring by assigning each t_i the degree of 1 such that

$$\mathcal{SP}^* = (\mathcal{SP}^1, \mathcal{SP}^2, \mathcal{SP}^3, \ldots)$$

is a graded ring with each $\sigma_k(t_1, \ldots, t_n)$ having degree k. In particular, each $\mathcal{SP}^k$ is an additive group of (homogeneous) symmetric polynomials with a basis

$$\mathcal{B}_{\mathcal{SP}^k} = \left\{ \prod_{i_j \in I} \sigma_{i_j}(t_1, \ldots, t_n) \mid I = i_1, \ldots, i_r \text{ is a partition of } k \right\}.$$

Instead of dragging the discussion further without motivation, we present our end goal to provide context for the rest of this section. Ultimately, we claim the following.

Theorem 5.4.8. *The graded ring $\mathcal{SP}^* \cong \mathrm{H}^*(\mathrm{Gr}_n(\mathbb{C}^\infty); \mathbb{Z})$.*

In order to show this, we need to construct an equivalent basis (generating set) for $\mathcal{SP}^k$ while also showing that it generates $\mathrm{H}^{2k}(\mathrm{Gr}_n(\mathbb{C}^\infty); \mathbb{Z})$. To do this, consider the following definition.

Definition 5.4.9. Let $I = i_1, \ldots, i_r$ be a partition for k. The ***k-th Newtonian power sum*** is defined

$$s_I(\sigma_1(t_1, \ldots, t_n)), \ldots, \sigma_k(t_1, \ldots, t_n)) = t_1^{i_1} + \ldots t_r^{i_r}.$$

Thankfully, the Newtonian power sum is easy to compute using the following identity.

Theorem 5.4.10 (Newton's Formula for Power Sums)**.** *Assuming the above definitions and terminology, the summation*

$$\sum_{i=0}^{n} (-1)^{n-i} \sigma_{n-i}(t_1, \ldots, t_n) s_i(\sigma_1(t_1), \ldots, \sigma_i(t_n)) = 0.$$

Proof. Using the convention that $\sigma_0 = 1$, we see that

$$\sum_{i=0}^{n} \sigma_i(t_1, \ldots, t_n) = \prod_{i=1}^{n} (1 + t_i),$$

which follows by factoring the terms on the left hand side. Generalizing this idea shows that

$$\sum_{i=0}^{n} x_i \sigma_i(t_1, \ldots, t_n) = \prod_{i=1}^{n} (x_i + t_i),$$

and we use this identity to prove the claim. By substituting $x_i = -t_i$, we see that we have

$$\begin{aligned}\sum_{i=0}^{n}(-1)^{n-i}\sigma_{n-i}(t_1,\dots,t_n)s_i(\sigma_1(t_1),\dots,\sigma_i(t_n)) &= (-1)^n\sum_{i=1}^{n}\left(\sum_{j=0}^{n}(-t_i)^j\sigma_{n-i}(t_1,\dots,t_n)\right)\\ &= (-1)^n\sum_{i=1}^{n}\left(\prod_{j=0}^{n}(-t_i+t_j)\right)\end{aligned}$$

which vanishes, since at some point we would have $i = j$. ■

The above theorem shows how we can recursively compute s_I, and it directly follows that the $p(k)$-many $s_I(\sigma_1(t_1,\dots,t_n)),\dots,\sigma_k(t_1,\dots,t_n))$ also form a basis for $\mathcal{SP}^k$. Now, let ω be a complex $\mathbb{C}^n$-bundle that splits as a Whitney sum

$$\omega = \bigoplus_{i=1}^{n}\eta_i = \eta_1\oplus\dots\oplus\eta_n.$$

Then applying Theorem 5.3.4 shows that

$$1+\mathrm{c}_1(\omega)+\mathrm{c}_2(\omega)+\dots+\mathrm{c}_n(\omega) = \prod_{i=1}^{n}(1+\mathrm{c}_i(\eta_i)).$$

We have seen this before. Using the above discusion, we see that this implies that

$$\sigma_i(\mathrm{c}_1(\eta_1),\dots,\mathrm{c}_i(\eta_i)) = \mathrm{c}_i(\omega).$$

However, recall the equivalence of bases

$$s_I(\sigma_1,\dots,\sigma_k) = \mathcal{B}_{\mathcal{SP}^k} = \left\{\prod_{i_j\in I}\sigma_{i_j}(t_1,\dots,t_n) \mid I = i_1,\dots,i_r \text{ is a partition of } k\right\}.$$

As mentioned before, the product of Chern classes

$$\mathrm{c}_I(\gamma^n) = \prod_{i_a\in I}\mathrm{c}_{i_a}(\gamma^n)$$

already forms a basis of $\mathrm{H}^{2k}(\mathrm{Gr}_n(\mathbb{C}^\infty);\mathbb{Z})$, but this is precisely $\mathcal{B}_{\mathcal{SP}^k}$ with $\sigma_{i_j} = \mathrm{c}_{i_a}(\gamma^n)$, so we see that $s_I(\mathrm{c}_1(\omega),\dots,\mathrm{c}_k(\omega))$ is an alternative basis for $\mathrm{H}^{2k}(\mathrm{Gr}_n(\mathbb{C}^\infty);\mathbb{Z})$. This implies Theorem 5.4.8.

Definition 5.4.11. Let M be a compact complex n-dimensional manifold. For each partition I of n, the notation $s_I[M]$, will be used in replacement of the product $\langle s_I(\mathrm{c}(\tau_M)),\mu_{2n}\rangle$.

In order to prove our final result for Chern classes, we need the following result from Thom, which we state without proof since its proof is not relevant (for a proof, see [MS74]).

Theorem 5.4.12 (Thom)**.** *The characteristic class $s_I(\mathrm{c}(\omega\oplus\omega'))$ of a Whitney sum is equivalent to the sum*

$$\sum_{IJ=1}s_I(\mathrm{c}(\omega))s_J(\mathrm{c}(\omega')).$$

In other words, we are able to split a Whitney sum into a product of Newtonian power sums. Our final stepping stone to the last result of this section is the following.

Theorem 5.4.13. *Let M and N be compact complex n and m dimensional manifolds respectively. Then the number*

$$s_I[M \times N] = \sum_{IJ=1} s_I[M] s_J[N].$$

Proof. The Cartesian product of the tangent bundles of $M \times N$

$$\tau_M \times \tau_N \cong (\pi_M^* \tau_M) \oplus (\pi_N^* \tau_N)$$

where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$. Hence, referencing the definition above and Theorem 5.4.12 shows that

$$\begin{aligned} s_I[M \times N] &= \langle s_I(\tau_M \times \tau_N), \mu_{2n} \times \mu_{2m} \rangle \\ &= \langle s_I((\pi_M^* \tau_M) \oplus (\pi_N^* \tau_N)), \mu_{2n} \times \mu_{2m} \rangle \\ &= \left\langle \sum_{IJ=1} s_I(\mathrm{c}(\tau_M)) s_J(\mathrm{c}(\tau_N)), \mu_{2n} \times \mu_{2m} \right\rangle \\ &= \sum_{IJ=1} s_I[M] s_J[N] \end{aligned}$$

which is what we wanted to prove. ■

We have finally arrived at our destination for this section. We state it as follows.

Theorem 5.4.14. *Let $M_1, \ldots, M_n$ be complex manifolds with $s_k[M_k] \neq 0$. Then the $p(n) \times p(n)$ matrix*

$$\mathcal{M} = \left(\mathrm{c}_{i_1} \cdots \mathrm{c}_{i_r} [M_{j_1} \times \ldots \times M_{j_s}] \right),$$

where $I = i_1, \ldots, i_r$ and $J = j_1, \ldots, j_s$ are partitions of n, is nonsingular.

Remark 5.4.15. An identical result holds for Pontryagin numbers, which we talk about in the next section.

Proof. This is precisely why we went through all of that trouble with redefining bases. Instead of using the Chern numbers themselves, which would make this proof a nightmare, we use linear combinations of $s_I(\mathrm{c})$. Generalizing Theorem 5.4.13 to multiple partitions, we see that

$$s_I[M_{j_1} \times \ldots \times M_{j_p}] = \sum_{I_1 \cdots I_p = 1} s_{I_1}[M_{j_1}] \cdots s_{I_p}[M_{j_p}].$$

Note that $s_I[M_{j_1} \times \ldots \times M_{j_p}] = 0$ only when $r \geq p$, so this matrix is lower triangular if the partitions are arranged in a sufficient order. Applying a special case of Theorem 5.4.13 shows that each entry in the diagonal can be written as a nonzero product, and the result follows. ■

6 Pontryagin Classes

We have now arrived at our final characteristic class, the Pontryagin class. Not surprisingly, defining these characteristic classes requires all of the previously derived theory, but we hope that the reader can appreciate the reward of this work.

6.1 Pontryagin Classes

Consider the following (familiar) definition.

Definition 6.1.1. Let V be a real vector space, and let $V \otimes \mathbb{C} = V \otimes_{\mathbb{R}} \mathbb{C}$ denote the ***complexification*** of V. In a way, one can say that complexifying each fiber F of the real $\mathbb{R}^n$-bundle ξ yields a complex $\mathbb{C}^n$-bundle with generic fibers $F \otimes \mathbb{C}$ over the same base space. We will denote this new bundle as $\xi \otimes \mathbb{C}$ and we will call it the ***complexification of the real vector bundle*** ξ.

We begin by describing what the elements in $F \otimes \mathbb{C}$ look like. They can be decomposed (uniquely) as a sum $x + iy$ where $x, y \in F$. This yields a direct sum expression

$$F \otimes \mathbb{C} = F \oplus iF,$$

which means that the underlying real vector bundle $(\xi \otimes \mathbb{C})_{\mathbb{R}}$ is canonically isomorphic to the Whitney sum $\xi \oplus \xi$.

Theorem 6.1.2. *The complexification $\xi \otimes \mathbb{C}$ of a real vector bundle is isomorphic to its own conjugate bundle $\overline{\xi \otimes \mathbb{C}}$.*

The correspondence $f : x + iy \mapsto x - iy$ maps the total space $E(\xi \otimes \mathbb{C})$ homeomorphically onto itself, and is $\mathbb{R}$-linear where each fiber $f(i(x + iy))$ is equivalent to $-if(x + iy)$. In fact, let us consider the total Chern class

$$\mathrm{c}(\xi \otimes \mathbb{C}) = 1 + \mathrm{c}_1(\xi \otimes \mathbb{C}) + \mathrm{c}_2(\xi \otimes \mathbb{C}) + \cdots + \mathrm{c}_n(\xi \otimes \mathbb{C})$$

of this complexified $\mathbb{R}^n$-bundle. If we set this equal to

$$\mathrm{c}(\overline{\xi \otimes \mathbb{C}}) = 1 - \mathrm{c}_1(\xi \otimes \mathbb{C}) + \mathrm{c}_2(\xi \otimes \mathbb{C}) - \cdots \pm \mathrm{c}_n(\xi \otimes \mathbb{C}),$$

we see that all of the odd Chern classes

$$\mathrm{c}_1(\xi \otimes \mathbb{C}), \mathrm{c}_3(\xi \otimes \mathbb{C}), \cdots$$

are all elements of order 2. This leads us to defining our final characteristic class.

Definition 6.1.3. Ignoring all of the pesky elements of order 2, the ***i-th Pontryagin class*** is defined to be the (integral) cohomology class $(-1)^i \mathrm{c}_{2_i}(\xi \otimes \mathbb{C})$, which we write as $\mathrm{p}_i(\xi) \in \mathrm{H}^{4i}(B(\xi); \mathbb{Z})$

Remark 6.1.4. By definition, $\mathrm{p}_i(\xi)$ is zero for $i > n/2$. The total Pontryagin class is defined to be the unit

$$\mathrm{p}(xi) = 1 + \mathrm{p}_1(\xi) + \cdots + \mathrm{p}_{\lfloor n/2 \rfloor}(\xi)$$

in the ring $H^{\Pi}(B; \mathbb{Z})$.

We state the following naturality result without proof (see [MS74]).

Theorem 6.1.5. *Pontryagin classes are natural with respect to bundle maps. Also, if ε_B^k is a trivial $\mathbb{R}^k$-bundle, then* $\mathrm{p}\left(\xi \oplus \varepsilon_B^k\right) = \mathrm{p}(\xi)$.

Just like all of the previously-discussed characteristic classes, we would like the Pontryagin class to satisfy a ”product formula”. However, since the odd Chern classes of $\xi \otimes \mathbb{C}$ have been discarded, the best we can do is the following.

Theorem 6.1.6. *The total Pontryagin class* $\mathrm{p}(\xi \oplus \eta)$ *of a Whitney sum is congruent to* $\mathrm{p}(\xi)\mathrm{p}(\eta)$ *modulo elements of order* 2. *That is to say,*

$$2(\mathrm{p}(\xi \oplus \eta) - \mathrm{p}(\xi)\mathrm{p}(\eta)) = 0.$$

Proof. Since $(\xi \oplus \eta) \otimes \mathbb{C}$ is isomorphic to $(\xi \otimes \mathbb{C}) \oplus (\eta \otimes \mathbb{C})$ we have

$$\mathrm{c}_k((\xi \oplus \eta) \otimes \mathbb{C}) = \sum_{i+j=k} \mathrm{c}_i(\xi \otimes \mathbb{C})\mathrm{c}_j(\eta \otimes \mathbb{C}).$$

Ignoring the odd Chern classes, which are all elements of order 2, it follows that

$$\mathrm{c}_{2k}((\xi \oplus \eta) \otimes \mathbb{C}) = \sum_{i+j=k} \mathrm{c}_{2i}(\xi \otimes \mathbb{C})\mathrm{c}_{2j}(\eta \otimes \mathbb{C})$$

modulo elements of order 2. Multiplying both sides of this congruence by $(-1)^k = (-1)^i(-1)^j$, it follows that

$$\mathrm{p}_k(\xi \oplus \eta) = \sum_{i+j=k} \mathrm{p}_i(\xi)\mathrm{p}_j(\eta)$$

as required. ∎

For any $\mathbb{C}^n$-bundle ω, the Chern classes $\mathrm{c}_i(\omega)$ determine the Pontryagin classes $\mathrm{p}_k(\omega_{\mathbb{R}})$ by the formula

$$1 - \mathrm{p}_1 + \mathrm{p}_2 - \cdots \pm \mathrm{p}_n = (1 - \mathrm{c}_1 + \mathrm{c}_2 - \cdots \pm \mathrm{c}_n)(1 + \mathrm{c}_1 + \mathrm{c}_2 + \cdots + \mathrm{c}_n).$$

Thus, $\mathrm{p}_k(\omega_{\mathbb{R}})$ is equal to

$$\mathrm{c}_k(\omega)^2 - 2\mathrm{c}_{k-1}(\omega)\mathrm{c}_{k+1}(\omega) + \cdots \pm 2\mathrm{c}_1(\omega)\mathrm{c}_{2k-1}(\omega) \mp 2\mathrm{c}_{2k}(\omega).$$

Example. Here is an interesting combinatorial example. Let τ be the tangent bundle of n-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$. Since the total Chern class $c(\tau)$ equals $(1+a)^{n+1}$, it follows that the Pontryagin classes $\mathrm{p}_k(\tau_{\mathbb{R}})$ are given by

$$\begin{aligned}(1 - \mathrm{p}_1 + \cdots \pm \mathrm{p}_n) &= (1 - \mathrm{c}_1 + \cdots \pm \mathrm{c}_n)(1 + \mathrm{c}_1 + \cdots + \mathrm{c}_n) \\ &= (1-a)^{n+1}(1+a)^{n+1} = \left(1 - a^2\right)^{n+1}.\end{aligned}$$

Therefore the total Pontryagin class $1 + \mathrm{p}_1 + \cdots + \mathrm{p}_n$ is equal to $\left(1+a^2\right)^{n+1}$. In other words

$$\mathrm{p}_k\left(\mathbb{P}^n(\mathbb{C})\right) = \binom{n+1}{k} a^{2k}$$

for $1 \leq k \leq n/2$, and the higher Pontryagin classes are zero since $\mathrm{H}^{4k}\left(\mathbb{P}^n(\mathbb{C})\right)$ for $k > n/2$.

Remark 6.1.7. Note that there is a slight amount of notational abuse here. In this example we are abbreviating the tangential Pontryagin class $\mathrm{p}_k\left(\tau(M)_{\mathbb{R}}\right)$ as $\mathrm{p}_k(M)$ for simplicity.

Now suppose we start with an oriented $\mathbb{R}^{2n}$-bundle ξ. If we complexify and pass to the underlying real vector bundle, we obtain a $\mathbb{R}^{2n}$-bundle $(\xi \otimes \mathbb{C})_{\mathbb{R}}$ with a preferred orientation.

Theorem 6.1.8. *The $\mathbb{R}^{2n}$-bundle $(\xi \otimes \mathbb{C})_{\mathbb{R}} \cong \xi \oplus \xi$ under an isomorphism which either preserves or reverses orientation according as $n(n-1)/2$ is even or odd.*

Proof. Let $v_1, \cdots, v_n$ be an ordered basis for a typical fiber F of ξ. Then the vectors $v_1, iv_1, \cdots, v_n, iv_n$ form an ordered basis determining the preferred orientation for $(F \otimes \mathbb{C})_{\mathbb{R}}$. Identifying this with the real direct sum $F \oplus iF \cong F \oplus F$, the basis $v_1, \cdots, v_n$ for F followed by the basis $iv_1, \cdots, iv_n$ for iF gives a different ordered basis. Evidently the permutation which transforms one ordered basis to the other has $\mathrm{sgn}(-1)^{(n-1)+(n-2)+\cdots+1} = (-1)^{n(n-1)/2}$. ■

Using this theorem, we can establish a very intersting result that unites Pontryagin and Euler classes. This should not come as a surprise, however; because characteristic classes are so interconnected theoretically, there are plenty of areas of crossover. This just happens to be one of them.

Corollary 6.1.9. *If ξ is an oriented $\mathbb{R}^{2k}$-plane bundle, then the Pontryagin class $\mathrm{p}_k(\xi)$ is equal to the square of the Euler class $\mathrm{e}(\xi)$.*

Proof. Because $\mathrm{p}_k(\xi) = (-1)^k \mathrm{c}_{2k}(\xi \otimes \mathbb{C}) = (-1)^k \mathrm{e}\left((\xi \otimes \mathbb{C})_{\mathbb{R}}\right)$, the previous result and Theorem 4.2.10 imply that the Euler class

$$\mathrm{e}\left((\xi \otimes \mathbb{C})_{\mathbb{R}}\right) = \mathrm{e}(\xi \oplus \xi) = \mathrm{e}(\xi)^2$$

multiplied by the $\mathrm{sign}(-1)^{2k(2k-1)/2} = (-1)^k$. ■

6.2 Cohomology of the Oriented Grassmann Manifold

In this section we make brief remarks about applying the ideas of orientation to $\mathrm{Gr}_n(\mathbb{R}^{n+k})$, and looking at their cohomology.

Definition 6.2.1. The ***oriented Grassmann manifold*** $\widetilde{\mathrm{Gr}_n}(\mathbb{R}^{n+k})$ is the quotient space consisting of all oriented n-planes in $(n+k)$-space, which are oriented as vector spaces.

Note that $\widetilde{\mathrm{Gr}_n}(\mathbb{R}^{n+k})$ is a compact CW–complex of dimension nk. Passing to the direct limit we obtain an infinite CW-complex $\widetilde{\mathrm{Gr}_n} = \widetilde{\mathrm{Gr}_n}(\mathbb{R}^\infty)$. Some authors like to use the notation $\mathrm{BSO}(n)$ and $\mathrm{BO}(n)$ to denote $\widetilde{\mathrm{Gr}_n}$ and $\widetilde{\mathrm{Gr}_n}(\mathbb{R}^{n+k})$ respectively, and we use both notations interchangeably.

Definition 6.2.2. The ***oriented universal bundle*** $\widetilde{\gamma}^n$ is the bundle obtained by lifting γ^n over Gr_n to an oriented $\mathbb{R}^n$-bundle over $\widetilde{\mathrm{Gr}_n}$.

In order to observe the cohomology of oriented Grassmann manifolds, we need to specify the coefficients in our cohomology class. We will observe its cohomology with coefficients in a commutative ring Λ containing $\frac{1}{2}$. The motivation for choosing such a specific coefficient domain is because this nullifies 2-torsion, allowing us to use Pontryagin classes.

Example. An immediate example of such a domain Λ is the ring $\mathbb{Z}[1/2]$ of rational integers. However, our arguments will work equally well with coefficients in the field of rational numbers $\mathbb{Q}$, or in any field of characteristic other than 2. The result will be only slightly more complicated than the cases of cohomology mod 2, but there is no need to worry about this.

Theorem 6.2.3. *If Λ is a commutative ring containing $\frac{1}{2}$, then the cohomology ring $\mathrm{H}^*\left(\widetilde{\mathrm{Gr}}_{2m+1};\Lambda\right)$ is a polynomial ring over Λ generated by the Pontryagin classes*

$$\mathrm{p}_1\left(\widetilde{\gamma}^{2m+1}\right),\ldots,\mathrm{p}_m\left(\widetilde{\gamma}^{2m+1}\right).$$

In a similar way, $\mathrm{H}^*\left(\widetilde{\mathrm{Gr}}_{2m};\Lambda\right)$ is a polynomial ring over Λ generated by the Pontryagin classes $\mathrm{p}_1\left(\gamma^{2m}\right),\ldots,\mathrm{p}_{m-1}\left(\gamma^{2m}\right)$ and the Euler class $\mathrm{e}\left(\widetilde{\gamma}^{2m}\right)$.

6.3 Pontryagin Numbers and Oriented Cobordism

Just as we defined Chern numbers using integer partitions, we will define the so-called Pontryagin numbers in a similar way. This provides us with sufficient machinery to delve into the theory of cobordism, and how the characteristic classes and numbers that we have defined in this paper yield information about cobordism problems.

Definition 6.3.1. Let M be a $4n$-dimensional smooth compact and oriented manifold. For each partition $I = i_1,\ldots,i_r$ of the dimension n, the ***I-th Pontryagin number*** is the integer

$$\mathrm{p}_I[M] = \mathrm{p}_{i_1}\cdots\mathrm{p}_{i_r}[M] = \langle \mathrm{p}_{i_1}(\tau_M)\cdots\mathrm{p}_{i_r}(\tau_M),\mu_{4n}\rangle.$$

Remark 6.3.2. If we change the orientation of M, the Pontryagin class does not change. However, the fundamental homology class μ_{4n} *does* change sign, so the Pontryagin number changes sign as well. Thus, if the Pontryagin number is nonzero, then M cannot possess any orientation reversing diffeomorphism.

Finally, we present C.T.C. Wall's deep result (see [Wal16] for a proof) that solves, as we saw with Stiefel-Whitney numbers and unoriented cobordism, the problem of oriented cobordism using Stiefel-Whitney numbers alongside Pontryagin numbers.

Theorem 6.3.3 (Wall)**.** *Stiefel-Whitney and Pontryagin numbers completely classify closed manifolds up to oriented cobordism. That is, two (sufficiently defined) manifolds are cobordant if and only if their Stiefel Whitney and Pontryagin numbers are equivalent to each other respectively.*

7 Cobordism

It is at this point where we arrive at our destination. As we saw in Chapters 2 and 6, unoriented and oriented cobordism problems can be solved elegantly using characteristic classes and numbers. In this final chapter, we will see how we can evaluate cobordism groups and rings.

7.1 Cobordism Groups and Rings

While we have previously defined cobordism in cases where we need specific assumptions in order to use characteristic classes, the definition we state below will be the one we use from here on out.

Definition 7.1.1. Let M and N be two smooth n-dimensional compact oriented manifolds. We say that the M and N are ***cobordant*** (or belong to the same ***cobordism class***) if there exists an $(n+1)$-dimensional manifold W with boundary (with the same conditions of M and N) such that ∂W is diffeomorphic to $M \sqcup \overline{N}$ as an oriented n-manifold. We call W the ***cobordism*** of M and N in this case, and when M and N are cobordant we write $M \Cap N$ when W is irrelevant.

Remark 7.1.2. Note that the notation $\overline{N}$ denotes a manifold with the opposite orientation of N. Also, certain authors such as [Hir76] prefer to define cobordism slightly differently, but we will not use their conventions.

It is immediate from the above definition that all diffeomorphic manifolds are cobordant to each other. However, an idea that we have *not* seen with cobordisms yet is how we can associate algebraic objects to them. It turns out that the following holds.

Theorem 7.1.3. *Cobordism forms an equivalence relation.*

Proof. To prove reflexivity, let M be a manifold with the above assumptions. Then $M \times [0,1]$ is a cobordism of M and M; just note that $\partial(M \times [0,1]) \cong (M \cup \{0\}) \sqcup (M \cup \{1\}) \cong M \sqcup \overline{M}$. Thus, $M \Cap M$ in this case. To show symmetry, let W be a cobordism of M and N such that $M \sqcup \overline{N} \cong \partial W$. Then $\overline{W}$ is a cobordism for N and M, since we clearly have $N \sqcup \overline{M} = \partial \overline{W}$. Hence, if $M \Cap N$ then $N \Cap M$. To show transitivity, let $M \Cap N$ and $N \Cap P$ such that $M \sqcup \overline{N} = \partial W$ and $N \sqcup \overline{P} = \partial V$. Then the union $W \cup V$ is a cobordism for M and P, so $M \Cap P$. ■

Using this equivalence relation, we can start creating algebraic objects from cobordisms. We will do it for the oriented case (the unoriented case is similar, and we leave it as a remark).

Definition 7.1.4. The ***oriented cobordism group*** $(\Omega_n, \sqcup)$ is the group of n-dimensional oriented manifolds up to cobordism, modulo the equivalence relation of cobordism. Note the following:

- (Identity) The identity element in Ω_n is just the equivalence class of $\oslash$.
- (Inverses) Every manifold $M \in \Omega_n$ has an inverse $\overline{M}$, and the cobordism of M and $\overline{M}$ is the boundary of $M \times [0,1] \Cap \oslash$.
- (Associativity) This follows from the properties of $\sqcup$.

Remark 7.1.5. Note that the commutative group operation $\sqcup$ makes Ω_n into an abelian group. Additionally, some authors use the notation $\mathfrak{N}^n$ to denote the ***unoriented cobordism group***, which has the same properties as the above. However, we will not focus on $\mathfrak{N}^n$ as much in this work.

Since we are working with algebraic objects instead of topological ones, it would help to compute some of them for small values of n. While computing Ω_n for $n \geq 2$ involves complicated techniques from surgery theory, we can compute Ω_n for $n = 1, 2$ fairly easily. We rely on the classification of 1 and 2-manifolds to help us.

Example. The most beautiful example is the computation of Ω_1. Note that the only compact (boundaryless) 1-manifold is the circle $\mathbb{S}^1$ (or alternatively that every 1-manifold is the disjoint union of circles), so it suffices to show that $\mathbb{S}^1 \Cap \oslash$. Consider the following diagram.

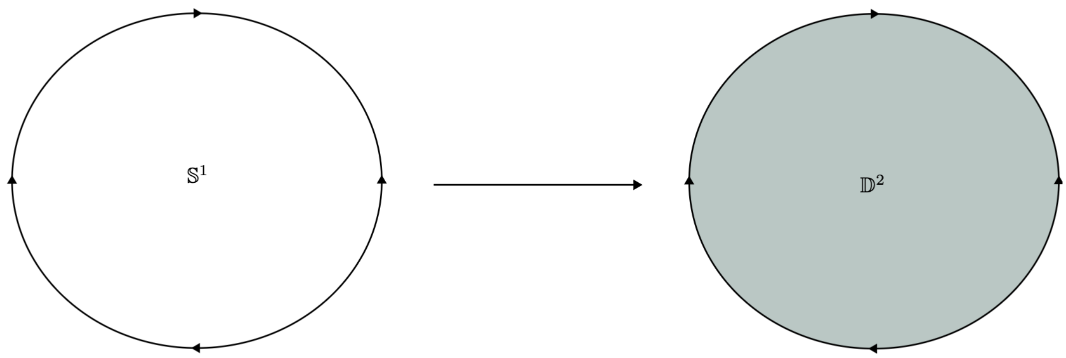


Figure 7.1. A visual depiction of why $\mathbb{S}^1 \cong \partial\mathbb{D}^2$, implying that $\mathbb{S}^1 \Cap \oslash$.

Hence, we see that $\Omega_1 = 0$ by considering Figure 7.1. A similar computation can be made for Ω_2, since all 2-manifolds are properly classified by their genus. All of these are cobordant to $\oslash$, since they are the boundaries of handlebodies, so we see that $\Omega_2 = 0$.

Because $\sqcup$ is a commutative operation, we can transform cobordism groups into rings.

Definition 7.1.6. Let M and N be n and m-dimensional manifolds with the above assumptions respectively. Taking their Cartesian product $M \times N$ gives rise to an associative and bilinear product $\Omega_n \times \Omega_m \to \Omega_{n+m}$. Thus, the sequence

$$\Omega_* = (\Omega_0, \Omega_1, \Omega_2, \dots) = \bigoplus_{n=0}^{\infty} \Omega_n$$

has the structure of a graded ring with a two-sided identity $1 \in \Omega_0$, and it is a commutative ring since

$$M \times N \cong (-1)^{mn} N \times M$$

as oriented manifolds. We call this commutative graded ring Ω_* the ***oriented cobordism ring***.

A natural question is whether Ω_* is computable. While the answer is yes (see [Sto68]), the result is quite involved. While not presenting the full computation of Ω_*, we do show a nice relation of Ω_* at the end of this chapter.

7.2 A Precursor to Thom's Theory

The heart of Thom's theory is about computing Ω_n and Ω_* using homotopy theory. Before we get into this computation, some definitions are in order.

Definition 7.2.1. Let ξ be an $\mathbb{R}^k$-bundle with a Euclidean metric, and let $P \subset E(\xi)$ be the subset $P = \{v \in E(\xi) \ : \ |v| \geq 1\}$. Then the quotient space $E(\xi)/P$ (where P is "pinched" to a single point) is called the ***Thom space*** $\mathrm{Th}(\xi)$. The Thom space has a specific base point which we denote as t_0 such that

$$\mathrm{Th}(\xi)\backslash\{t_0\} = \{v \in E(\xi) \ : \ |v| < 1\}.$$

Furthermore, if the base space of ξ is compact, then we define $\mathrm{Th}(\xi)$ to be the Alexandroff compactification of $E(\xi)$, such that $\mathrm{Th}(\xi) = E(\xi) \cup \{\infty\}$. In this case our base point t_0 would be some point at infinity.

Remark 7.2.2. If the base space B is a CW–complex, then $\mathrm{Th}(\xi)$ is a $(k-1)$–connected CW–complex possessing one $(n+k)$–cell corresponding to each n–cell of B. For a proof, see [MS74].

Our primary focus at the moment is on the homology groups of $\mathrm{Th}(\xi)$, since that will naturally lead to homotopy-theoretic ideas.

Theorem 7.2.3. *If ξ is an oriented $\mathbb{R}^k$-bundle over B, then* $\mathrm{H}_{k+i}(\mathrm{Th}(\xi), t_0; \mathbb{Z}) \cong \mathrm{H}_i(B(\xi); \mathbb{Z})$.

Proof. Note that from Definition 2.1.4 that B is embedded as the zero section of the space $E\backslash P \cong \mathrm{Th}(\xi)\backslash\{t_0\}$. Let $\mathrm{Th}_0(\xi) = E_0/P$ be the compliment of the zero section in $\mathrm{Th}(\xi)$. The compliment $\mathrm{Th}_0(\xi)$ is contractible, which means that the long homology sequence of the triple $(\mathrm{Th}(\xi), \mathrm{Th}_0, t_0)$

$$\longrightarrow \mathrm{H}_n(\mathrm{Th}_0(\xi), t_0; \mathbb{Z}) \longrightarrow \mathrm{H}_n(\mathrm{Th}(\xi), t_0; \mathbb{Z}) \longrightarrow \mathrm{H}_n(\mathrm{Th}(\xi), \mathrm{Th}_0(\xi); \mathbb{Z}) \xrightarrow{\partial} \mathrm{H}_{n-1}(\mathrm{Th}_0(\xi), t_0; \mathbb{Z}) \longrightarrow$$

is exact since $\mathrm{H}_n(\mathrm{Th}_0(\xi), t_0; \mathbb{Z})$ is trivial. This means that we have a canonical isomorphism

$$\mathrm{H}_n(\mathrm{Th}(\xi), t_0; \mathbb{Z}) \cong \mathrm{H}_n(\mathrm{Th}(\xi), \mathrm{Th}_0(\xi); \mathbb{Z}),$$

and by excision we have $\mathrm{H}_n(\mathrm{Th}(\xi), t_0; \mathbb{Z}) \cong \mathrm{H}_n(E(\xi), E_0; \mathbb{Z})$. The result follows from Theorem 4.2.6. ∎

Definition 7.2.4. The set $\mathfrak{A}_{\text{finite}}$ denotes the ***class of finite abelian groups***. A homomorphism $\varphi : G \to H$ for $G, H \in \mathfrak{A}_{\text{finite}}$ is called an ***$\mathfrak{A}_{\textit{finite}}$-isomorphism*** if $\ker(\varphi), \operatorname{coker}(\varphi) \in \mathfrak{A}_{\text{finite}}$.

In order to prove a key result in this section, we rely on the following theorem from Serre that we state without proof (see [MS74]), since its proof is not relevant to this work.

Theorem 7.2.5. *Let X be a finite complex which is $(k-1)$–connected for $k \geq 2$. Then the Hurewicz homomorphism $\pi_r(X) \to \mathrm{H}_r(X; \mathbb{Z})$ is an $\mathfrak{A}_{\textit{finite}}$-isomorphism for $r < 2k-1$.*

Using the above result, we can now prove a corollary which will be used later when computing the oriented cobordism group Ω_n.

Corollary 7.2.6. *If $\mathrm{Th}(\xi)$ is the Thom space of an oriented $\mathbb{R}^k$-bundle over the finite complex B, then there is an $\mathfrak{A}_{\textit{finite}}$-isomorphism $\pi_{n+k}(\mathrm{Th}(\xi), t_0) \to \mathrm{H}_n(B; \mathbb{Z})$ for all dimensions $n < k-1$.*

Proof. This follows from Theorem 7.2.3 and Theorem 7.2.5. ∎

7.3 Thom's Theory and Ending Cobordism

Given some oriented $\mathbb{R}^k$-bundle ξ, our next objective is to see how we can approximate a continuous map $f : \mathbb{S}^m \to \mathrm{Th}(\xi)$ via a smooth map. However, the Thom space $\mathrm{Th}(\xi)$ is not a manifold, so approximating f by a smooth map does not make sense. To patch this issue, we take the compliment $\mathrm{Th}(\xi)\backslash t_0$, which is a manifold. Doing so allows us to use a smooth map f_0 to approximate f, which coincides on the (open) subset $f^{-1}(t_0) = f_0^{-1}(t_0)$ and is smooth throughout $f_0^{-1}(\mathrm{Th}(\xi)\backslash t_0)$. Our first result (for a proof, see [MS74]) sets the stage for the rest of Thom's theory.

Theorem 7.3.1. *Let ξ be an oriented $\mathbb{R}^k$-bundle, and let $f : \mathbb{S}^m \to \mathrm{Th}(\xi)$ be continuous. Then f is homotopic to a smooth function f_0, which is smooth throughout $f_0^{-1}(\mathrm{Th}(\xi)\backslash t_0)$ and $f_0 \pitchfork B$, where B is the zero section (and the base space of ξ). Furthermore, the oriented cobordism class of the $(m-k)$-dimensional smooth manifold $f_0^{-1}(B)$ only depends on the homotopy class of f_0, and thus the correspondence $g \mapsto g^{-1}(B)$ gives rise to a group homomorphism*

$$\Phi_{\text{orient}} : \pi_m(\mathrm{Th}(\xi), t_0) \to \Omega_{m-k}$$

from the homotopy group of the Thom space to its corresponding oriented cobordism group.

While Theorem 7.3.1 establishes Φ_{orient} has a group homomorphism, we can do better if we restrict ξ to the oriented universal $\mathbb{R}^k$-bundle $\widetilde{\gamma}^k$ with base space $\widetilde{\mathrm{Gr}}_k(\mathbb{R}^\infty)$. Doing so gives the iconic result due to Thom, which states that this restriction gives rise to a group *isomorphism* instead.

Theorem 7.3.2 (Thom's Theorem)**.** *For $k > n+1$, there is a canonical group isomorphism*

$$\Phi_{\text{orient}} : \pi_{n+k}(\mathrm{Th}(\widetilde{\gamma}^k), t_0) \to \Omega_n.$$

In the unoriented case (i.e. taking γ^k), this also produces a group isomorphism

$$\Phi_{\text{orient}} : \pi_{n+k}(\mathrm{Th}(\gamma^k), t_0) \to \mathfrak{R}_n.$$

We will prove one part of this theorem, mainly the fact that Φ_{orient} is a surjective map. For a proof of injectivity, results can be found in [Hir76] and other works on the subject. We will use this (partial) result to prove other information regarding Ω_n as well.

Theorem 7.3.3. *Let $k \geq n$ and $p \geq n$. Then the group homomorphism*

$$\Phi_{\text{orient}} : \pi_{n+k}(\mathrm{Th}(\widetilde{\gamma_p}^k), t_0) \to \Omega_n$$

is a surjective map.

Proof. Let M be an n-dimensional compact oriented smooth manifold. By the Whitney Embedding Theorem, M can be embedded into $\mathbb{R}^{n+k}$, and by Theorem 2.2.6 we can choose an open neighborhood $U \subseteq \mathbb{R}^{n+k}$ of M that is diffeomorphic to $E(\nu^k)$ for some normal bundle ν^k. Using a Gauss map, we can map the total spaces $U \cong E(\nu^k) \mapsto E(\widetilde{\gamma_n}^k) \subseteq E(\widetilde{\gamma_p}^k)$, and composing this with $E(\widetilde{\gamma_p}^k) \to \mathrm{Th}(\widetilde{\gamma_p}^k)$ gives a smooth map $f_0 : U \to \mathrm{Th}(\widetilde{\gamma_p}^k)$ such that $f_0 \pitchfork B$ and $f_0^{-1}(B) = M$. In order to apply Theorem 7.3.1, we need to extend U to the sphere $\mathbb{S}^{n+k}$. To do this, we can extend f_0 to the Alexandroff compactification $\mathbb{R}^{n+k} \cup \{\infty\} \cong \mathbb{S}^{n+k}$ by forcing the map $\mathbb{S}^{n+k}\backslash U \mapsto t_0$. This gives a new map

$$\hat{f}_0 : \mathbb{S}^{n+k} \to \mathrm{Th}(\widetilde{\gamma_p}^k),$$

and by Theorem 7.3.1 this gives rise to the oriented cobordism class of M. ■

Using the above theorem, all of our work with characteristic classes pays off. Let $I = i_1, \ldots, i_r$ be a partition for k. We state the following result, which follows naturally from our work on Pontryagin classes and Pontryagin numbers in Chapter 6.

Theorem 7.3.4. *Let M be a $4k$-dimensional compact oriented smooth manifold. Then the map $M \mapsto \mathrm{p}_I[M]$ gives rise to a group homomorphism from Ω_{4k} to $\mathbb{Z}$, where $\mathbb{Z}$ is understood to be the group $(\mathbb{Z}, +)$.*

Proof. If M is the boundary of a $(4k+1)$-dimensional compact oriented smooth manifold with boundary, then we use the fact that $\mathrm{p}_I[M]$ vanishes. Otherwise, the identity $\mathrm{p}_I[M+M'] = \mathrm{p}_I[M] + \mathrm{p}_I[M']$ is clearly satisfied in the general sense, which completes the proof. ■

Before we continue onward on our path to our next result, we must recall a definition from Group Theory that is a common misconception.

Definition 7.3.5. Let G be an abelian group. The cardinality of a maximal linearly independent set in G is called the ***rank*** of G, which we denote as $\operatorname{rank}(G)$.

Using Theorem 5.4.14, we can first prove a lower bound on the rank of Ω_{4k}. Note that the notion of "rank" is with respect to Ω_{4k} as an abelian group.

Theorem 7.3.6. *The products of projective spaces*

$$\prod_{i_k \in I} \mathbb{P}^{2i_k}(\mathbb{C}) = \mathbb{P}^{2i_1}(\mathbb{C}) \times \mathbb{P}^{2i_2}(\mathbb{C}) \times \cdots \times \mathbb{P}^{2i_r}(\mathbb{C})$$

represent linearly independent elements of Ω_{4k}*, which means that* $\operatorname{rank}(\Omega_{4k}) \geq p(k)$ *where* $p(k)$ *is the number of partitions of* k*.*

Proof. This follows from the above discussion and Theorem 5.4.14. ■

While this is a nice result on its own, we can do better. We claim that the rank of Ω_{4k} is precisely $p(k)$, and we put an end to the oriented cobordism problem once and for all.

Theorem 7.3.7. *The oriented cobordism group* Ω_n *is finite for* $n \not\equiv 0 \pmod 4$*, and is finitely generated with* $\operatorname{rank}(\Omega_n) = p(k)$ *where* $n = 4k$*.*

Proof. By Theorem 7.3.3, Ω_n is the homomorphic image of the homotopy group $\pi_{n+k}(\mathrm{Th}(\widetilde{\gamma}_p^{\,k})), t_0)$ for $k \geq n$ and $p \geq n$, and by Corollary 7.2.6 we have that

$$\pi_{n+k}(\mathrm{Th}(\widetilde{\gamma}_p^{\,k}), t_0) \cong \mathrm{H}_n(\mathrm{Gr}_n(\mathbb{R}^\infty); \mathbb{Z})$$

as an $\mathfrak{A}_{\text{finite}}$-isomorphism. Using Theorem 6.2.3, the homology group $\mathrm{H}_n(\mathrm{Gr}_n(\mathbb{R}^\infty); \mathbb{Z})$ is finite for $n \not\equiv 0 \pmod 4$ with rank $p(k)$ for $n = 4k$. Hopping across maps, this implies that Ω_n is also finite for $n \not\equiv 0 \pmod 4$ and that $\operatorname{rank}(\Omega_n) \leq p(k)$ for $n = 4k$. By Theorem 7.3.6 we reach equality, which completes the proof. ■

All of this exposition has been on cobordism groups so far, however. What about the oriented cobordism ring Ω_*? While there are more involved techniques to directly compute it using spectra (see [Sto68] for details), there is an interesting result that immediately follows from the above discussion when we remove torsion by tensoring with the rationals.

Theorem 7.3.8. *Let* $I = i_1, i_2, \ldots, i_r$ *be a partition for* k*. Then*

$$\prod_{i_k \in I} \mathbb{P}^{2i_k}(\mathbb{C}) = \mathbb{P}^{2i_1}(\mathbb{C}) \times \mathbb{P}^{2i_2}(\mathbb{C}) \times \cdots \times \mathbb{P}^{2i_r}(\mathbb{C})$$

forms a basis of the vector space $\Omega_{4k} \otimes \mathbb{Q}$*. Thus,* $\Omega_{4k} \otimes \mathbb{Q}$ *is a polynomial algebra with (independent) generators* $\mathbb{P}^2(\mathbb{C}), \mathbb{P}^4(\mathbb{C}), \mathbb{P}^6(\mathbb{C}), \ldots$*.*

Bibliography

[Die08] Tammo tom Dieck. *Algebraic Topology.* European Mathematical Society, 2008.

[Hir76] Morris W. Hirsch. *Differential Topology*, volume 33. Springer Science + Business Media, 1976.

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*, volume 218. Springer Science + Business Media, 2013.

[MS74] John Milnor and James D. Stasheff. *Characteristic Classes*, volume 76. Princeton University Press, 1974.

[Spi05] Michael Spivak. *A Comprehensive Introduction to Differential Geometry*, volume 1. Publish or Perish Inc, 2005.

[Sto68] Robert E. Stong. *Notes on Cobordism Theory.* Princeton University Press, 1968.

[Wal16] Charles Terence Clegg Wall. *Differential Topology.* Cambridge University Press, 2016.