Sphere Eversion

Grace Howard

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A sphere, $\mathbb{S}^2$, can be turned inside-out without cutting, tearing, or creasing it. This process is know as *sphere eversion*. The halfway point of a sphere eversion, or an immersed sphere that is halfway inside-out (there is a symmetry interchanging the two sides of the surface), is called a *halfway model*.

An example of a halfway model is *Boy's surface*. It is an immersed projective plane, or a way of immersing a sphere such that antipodal points always map to the same place. This results in two opposite sheets of surface on top of each other. After that, if these sheets can be pulled apart and the surface can be simplified to a sphere with the correct side out, then pulling them apart the other way will result in an inside-out sphere. Another example is the *Morin surface*, a halfway model with four-fold rotational symmetry. A method of sphere eversion using halfway models is called *minimax eversion*.

An example

In this example, first there is an eversion of a twisted cylinder using ruled surfaces. Then, the cylinder is turned into a sphere.

Let $(\mathbf{r}, t) \in \mathbb{R}^3 \times \mathbb{R}$ where $\mathbf{r} = (x, y, z)$. Then, use the immersion of the cylinder $(h, \phi) \in \mathbb{R} \times \mathbb{S}^1$ given by

$$x = \sin(n-1)\phi - h\ \sin\phi$$

$$y = \cos(n-1)\phi + h\ \cos\phi$$

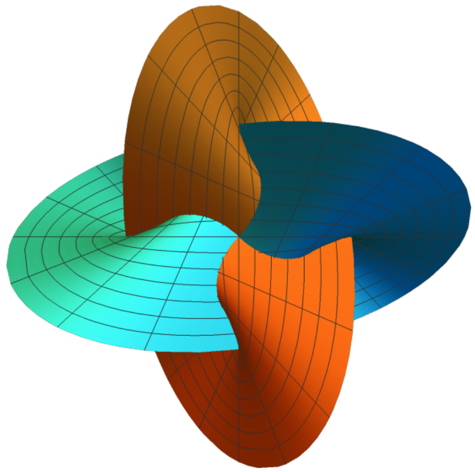
$$z = h\ \sin\ n\phi$$

with $n \in \mathbb{N}$, $n \geq 2$. This surface is smooth. Considering when $n = 2$,

$$x = \sin\phi - h\ \sin\phi$$

$$y = \cos\phi + h\ \cos$$

$$z = h\ \sin\ \phi.$$



All images are from *this paper*.

Then, $Q = (0,0,0)$, when

$$(h, \phi) = (1, \frac{\pi}{2})$$
$$(h, \phi) = (1, -\frac{\pi}{2})$$
$$(h, \phi) = (-1, 0)$$
$$(h, \phi) = (-1, \pi)$$

and has crossing of saddles when $(\pm\sqrt{2}, 0, 0)$ and $(0, \pm\sqrt{2}, 0)$ as well as lines of self-intersection at

$$x = z = 0$$

$$y = z = 0$$

and

$$x = \sqrt{2}\ \cos 2\varphi\ \cos\varphi$$
$$y = \sqrt{2}\ \cos 2\varphi\ \sin\varphi$$

$$z = -\frac{1}{2}\ \sin\ 4\varphi.$$

Generalizing this first to

$$x = t\ \cos\phi + \sin(n-1)\phi - h\ \sin\phi$$

$$y = t\ \sin\phi + \cos(n-1)\phi + h\ \cos\phi$$

$$z = h\ \sin\ n\phi - \frac{t}{n}\cos\ n\phi$$

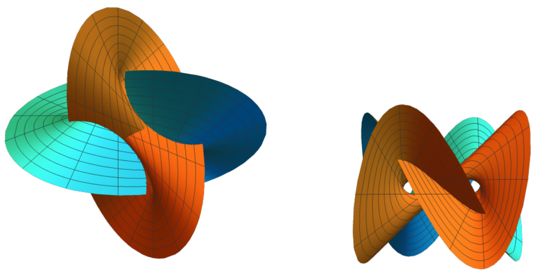
then to

$$x = t\ \cos\phi + p\ \sin(n-1)\phi - h\ \sin\phi$$

$$y = t\ \sin\phi + p\ \cos(n-1)\phi + h\ \cos\phi$$

$$z = h\ \sin\ n\phi - \frac{t}{n}\cos\ n\phi - qth$$

with $q \geq 0$ (this is smooth when $(n-1)p(1-q|t|) + qt^2 > 0$).



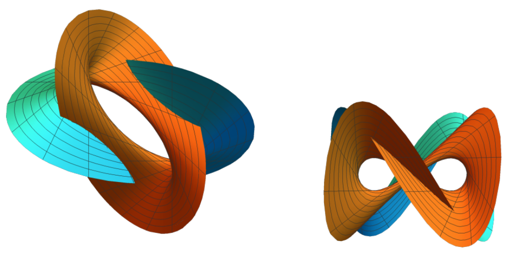
Starting from $|t| > 1$, $p = 1$, and $q = 0$ to end at $p = 0$, $qt = \pm 1$,

$$x = t\cos\phi - h\sin\phi$$

$$y = t\sin\phi + h\cos\phi$$

$$z = h\sin n\phi - (t/n)\cos n\phi \mp h$$

shown here for $n = 2$.



At $qt = \pm 1$ the intersection disappears at infinity. Let $p = 1 - |qt| \geq 0$. Then, parameterizing $\mathbb{S}^2$ by $\phi \in [-\pi, \pi]$ and $\theta \in [-\pi/2, \pi/2]$ using

$$X = R\cos\theta\cos\phi$$

$$Y = R\cos\theta\sin\phi$$

$$Z = R\sin\theta$$

to map onto the cylinder

$$\phi = \phi$$

$$h = \frac{\gamma \text{sin}\theta}{\text{cos}^n \theta}$$

with some $\gamma > 0$. The hole at infinity can be closed with stereographic projection. With that, the cylinder is mapped as

$$x' = x(\xi + \eta(x^2 + y^2))^{-\kappa}$$

$$y' = y(\xi + \eta(x^2 + y^2))^{-\kappa}$$

$$z' = \frac{z}{\xi + \eta(x^2 + y^2)}$$

with x, y, z defined by $x = t\ \text{cos}\phi + p\ \text{sin}(n-1)\phi - h\ \text{sin}\phi$, $\quad y = t\ \text{sin}\phi + p\ \text{cos}(n-1)\phi + h\ \text{cos}\phi$, $z = h\ \text{sin}\ n\phi - \frac{t}{n}\text{cos}\ n\phi - qth$, some $\xi, \eta \geq 0$,$|t| \leq 1$ keeping $\xi \geq 0$ with $\kappa = (n-1)/2n$. Then,

$$x'' = \frac{x' e^{\gamma z'}}{(\alpha + \beta(x'^2 + y'^2))}$$

$$y'' = \frac{y' e^{\gamma z'}}{(\alpha + \beta(x'^2 + y'^2))}$$

$$z'' = \frac{\alpha - \beta(x'^2 + y'^2)}{\alpha + \beta(x'^2 + y'^2)} \frac{e^{\gamma z'}}{\gamma} - \gamma^{-1} \frac{\alpha - \beta}{\alpha + \beta}$$

for $\alpha, \beta \geq 0$ and $\gamma = 2\sqrt{\alpha\beta}$ (this preserves inversion symmetry). When $\xi = 1$, $\eta = 0$, $\alpha = 1$, and $\beta \to 0$, this corresponds to the open hole. To close the hole, let $\eta > 0$, $\beta = 1$, $\alpha = 0$, $\xi = 0$, and $|t| > 1$. This means the inversion of the xy plane

$$x'' = \frac{x'}{x'^2 + y'^2}$$

$$y'' = \frac{y'}{x'^2 + y'^2}$$

$$z'' = -z'$$

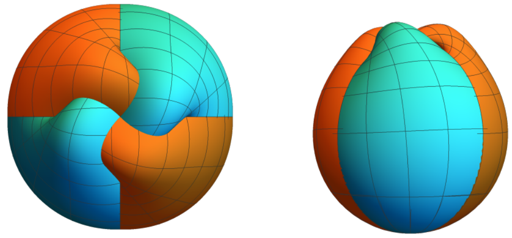
or

$$x'' = \frac{\eta^\kappa x}{(x^2 + y^2)^{1-\kappa}}$$

$$y'' = \frac{\eta^\kappa y}{(x^2 + y^2)^{1-\kappa}}$$

$$z'' = -\frac{z/\eta}{x^2 + y^2}$$

completes the inversion.



Thus, the full eversion map is

$$(\theta, \phi) \to (h, \phi) \to \mathbf{r} = (x, y, z) \to \mathbf{r}' = (x', y', z') \to \mathbf{r} = (x'', y'', z'').$$

After the inversion, z or z'' needs to be changed while keeping x, y or x'', y'' constant. With that, the sphere is still twisted, so let $\lambda \in [0, 1]$ and let

$$x = \frac{t(1 - \lambda + \lambda\cos^n\theta)\cos\phi - \lambda\omega\sin\theta\sin\phi}{\cos^n\theta}$$

$$y = \frac{t(1 - \lambda + \lambda\cos^n\theta)\sin\phi + \lambda\omega\sin\theta\sin\phi}{\cos^n\theta}$$

with

$$x'' = \eta^\kappa\cos\theta\frac{t(1 - \lambda + \lambda\cos^n\theta)\cos\phi - \lambda\omega\sin\theta\sin\phi}{(t^2(\lambda\cos^n\theta + (1 - \lambda))^2 + \lambda^2\omega^2\sin^2\theta)^{1-\kappa}}$$

$$y'' = \eta^\kappa\cos\theta\frac{t(1 - \lambda + \lambda\cos^n\theta)\sin\phi + \lambda\omega\sin\theta\sin\phi}{(t^2(\lambda\cos^n\theta + (1 - \lambda))^2 + \lambda^2\omega^2\sin^2\theta)^{1-\kappa}}$$

so that

$$x''^2 + y''^2 = \eta^{2\kappa}\cos^2\theta(t^2(\lambda\cos^n\theta + (1 - \lambda))^2 + \lambda^2\omega^2\sin^2\theta)^{-\frac{1}{n}}$$

is a function of $\cos^2\theta$. Then,

$$z = \lambda(\omega\sin\theta(\sin n\phi - qt)/\cos^n\theta - (t/n)\cos n\phi) - (1 - \lambda)\eta^{1+\kappa}t|t|^{2\kappa}\sin\theta/\cos^{2n}\theta.$$

Taking $\lambda = 0$ is the final sphere with radius $R = \eta^\kappa|t|^{-1/n}$.