INTEGRATION ON MANIFOLDS

EZRA FURTADO-TIWARI

Abstract.
In this paper, we introduce the machinery to define integration on manifolds. We first consider differential forms, and then understand how to integrate forms over manifolds.

1. Introduction

Manifolds allow us to abstractly define different kinds of surfaces and forms, which help us to rigorously generalize various properties to more abstract shapes. In particular, we will look into how integration can be generalized to arbitrary manifolds. Because of the definition of manifolds in terms of charts, we can define our operation over each individual chart in a parametrization of our manifolds.

As a result, one of the challenges is to make sure that the definition of integration is independent of the choice of charts used. To ensure that the definition holds regardless of the charts used, we define integration in terms of oriented charts.

2. Differential Forms

We first provide some useful definitions related to manifolds, so that we can develop the notion of differential forms.

Definition 2.1. Let M be a manifold. We define a *covector* to be a linear functional on the tangent space T_xM at a point $x \in M$. We define the *cotangent space* T_x^*X to be the vector space consisting of all covectors at a point $x \in M$. For an arbitrary vector space V, we define a covector to be a real-valued functional on V. The space of all covectors on V is denoted V^* and is called the *dual space* of V.

Definition 2.2. The *tangent bundle* TM of a manifold M is the disjoint unions of all tangent spaces at all points of M. Similarly, the *cotangent bundle* T^*M is the disjoint union of all cotangent spaces at all points of M.

We introduce the concept of multilinear algebra in order to understand the behavior of tensors and rigorously define differential forms.

Definition 2.3. Let $V_1, \ldots, V_k$ and W be finite-dimensional vector spaces. We say a function $f : V_1 \times \ldots \times V_k \to W$ is *multilinear* if it is linear in each variable. We write $L(V_1, \ldots, V_k; W)$ for the set of multilinear maps from $V_1 \times \ldots \times V_k$ to W.

Definition 2.4. Let $\mathcal{F}(V_1 \times \ldots \times V_k)$ be the *free vector space* formed by all finite linear combinations of k tuples $(v_1, \ldots, v_k)$ such that for $1 \leq i \leq n$, we have $v_i \in V_i$. Additionally, let $\mathcal{R}$ be the subspace of $\mathcal{F}(V_1, \ldots, V_k)$ spanned by vectors of the forms

$$(v_1, \ldots, v_i + v_i', \ldots, v_k) - (v_1, \ldots, v_i', \ldots, v_k) - (v_1, \ldots, v_i, \ldots, v_k)$$

Date: June 11, 2023.

and

$$(v_1, \ldots, av_i, \ldots, v_k) - a(v_1, \ldots, v_i, \ldots, v_k).$$

Definition 2.5. We define the *tensor product* of the spaces $V_1, \ldots, V_k$ to be the quotient space given by

$$V_1 \oplus \ldots \oplus V_k = \mathcal{F}(V_1 \times \ldots \times V_k)/\mathcal{R}.$$

We denote the equivalence class of an element $(v_1, \ldots, v_k)$ such that $v_i \in V_i$ for all $1 \leq i \leq k$ by $v_1 \oplus \ldots \oplus v_k = \Pi(v_1, \ldots, v_k)$, where $\Pi : \mathcal{F}(V_1, \ldots, V_k) \to V_1 \oplus \ldots \oplus V_k$ is the natural projection.

Note that we want the tensor product to satisfy linear properties, so we form an equivalence class over scalar multiplication and addition.

Proposition 2.6. *If $V_1, \ldots, V_k$ are finite-dimensional vector spaces, there is a uniquely-defined isomorphism*

$$V_1^* \oplus \ldots \oplus V_k^* \cong L(V_1, \ldots, V_k; \mathbb{R}).$$

Proof. Consider a map $\Phi : V_1 \oplus \ldots \oplus V_k \to L(V_1, \ldots, V_k; \mathbb{R})$ such that

$$\Phi(\omega_1, \ldots, \omega_k)(v_1, \ldots, v_k) = \omega_1(v_1) \ldots \omega_k(v_k).$$

$\Phi(\omega_1, \ldots, \omega_k)$ is linear in each variable v_i, so $\Phi(\omega_1, \ldots, \omega_k) \in L(V_1, \ldots, V_k; \mathbb{R})$. As a result, we can uniquely define a map $\tilde{\Phi}$ that takes $V_1^* \oplus \ldots \oplus V_k^*$ to $L(V_1, \ldots, V_k; \mathbb{R})$ satisfying

$$\tilde{\Phi}(\omega_1 \oplus \ldots \oplus \omega_k)(v_1, \ldots, v_k) = \omega_1(v_1) \ldots \omega_k(v_k).$$

then $\tilde{\Phi}$ takes the basis of $V_1^* \oplus \ldots \oplus V_k^*$ to the basis of $L(V_1, \ldots, V_k; \mathbb{R})$, so it is an isomorphism. ∎

This isomorphism allows us to use whichever definition is most convenient for each particular case.

Definition 2.7. Let V be a finite-dimensional vector space. A covariant k-tensor on V is an element of the product $\underbrace{V^* \oplus \ldots \oplus V^*}_{k \text{ times}}$. This is generally a real-valued function

$$t : \underbrace{V \times \ldots \times V}_{k \text{ times}} \to \mathbb{R}.$$

We let $T^k(V)$ be the vector space of covariant k-tensors on V, and we call k the rank of t for any $t \in T^k(V)$.

We denote the vector space of all k-covectors on a vector space V by $\Lambda^k(V^*)$.

Definition 2.8. We say a tensor is *symmetric* if it is invariant under any change of indices, and we say it is *antisymmetric* or *alternating* if it changes sign under any change of indices.

In other words, if an alternating tensor is permuted, it should be multiplied by the sign of the permutation.

Definition 2.9. We define the projection $\mathrm{Alt} : T^k(V^*) \to \Lambda^k(V^*)$ to satisfy

$$(\mathrm{Alt}\, \alpha)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\mathrm{sgn}\ \sigma) \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(n)}),$$

where S_k is the symmetric group of size k.

The idea of Alt is to create a map that takes non-alternating tensors to alternating tensors, and alternating tensors to themselves. By multiplying by the sign of the permutation for each permutation, we ensure that the result is alternating.

Definition 2.10 (Wedge Product)**.** We define the *wedge product* of $x \in \Lambda^k(V^*)$ and $y \in \Lambda^l(V^*)$ to be

$$x \wedge y = \frac{(k+l)!}{k!l!} \operatorname{Alt}(x \oplus y).$$

The wedge product preserves multilinearity, and the wedge product of two tensors on different dimensions is another tensor on the cross product of those dimensions, so we can use the wedge product to extend tensors to multiple different dimensions. This property will be useful in discussion about integration, as it will allow us to integrate independently in each chart.

Definition 2.11. Let M be a smooth n-manifold (possibly with boundary). Consider the subset

$$\Lambda^k T^* M = \bigsqcup_{p \in M} \Lambda^k(T_p^* M)$$

of the bundle of covariant k-tensors on M consisting of alternating tensors. We call a section of $\Lambda^k T^* M$ a *differential k-form*. We say k is the *degree* of the form.

The vector space of smooth differential k-forms is denoted by $\Omega^k(M)$. We can define integration naturally over differential forms on manifolds, which will allow us to extend our definition of integrals.

The final piece of machinery that we will need to develop before defining integration will be an operation called the pullback. This operation will allow us to define an analog to the antiderivative in terms of the differential.

Definition 2.12. We define a *rough vector field* to be an arbitrary map $X : TM \to M$ such that $x \mapsto X_p$ satisfying $X_p \in T_p M$.

Note that this definition is similar to that of a vector field, except the map is not necessarily continuous.

Definition 2.13. Let $F : M \to N$ be a smooth map between manifolds, and let $x \in M$ be a point. We call the linear map

$$dF_x^* : T_F^*(x) N \to T_x^* M$$

the *pullback* by F at x. If ω is a covector field of N, we say the *pullback of* ω *by* F is the rough vector field given by

$$(F^*\omega)_x = dF_x^*(\omega_{F(x)}).$$

3. Integration

In order to define integration on manifolds, we first define integration on simpler forms and then extend our definition to incorporate the various structures that we wish to integrate over.

Definition 3.1. A *domain of integration* is a bounded subset of $\mathbb{R}^n$ such that its boundary has n-dimensional measure zero.

Definition 3.2. Let $D \subset \mathbb{R}^n$ be a domain of integration, and ω be a continuous n-form on f. Then ω can be written as $f\, dx_1 \wedge \ldots \wedge dx_n$ for some continuous function $f : \overline{D} \to \mathbb{R}$. We define the integral of ω over U to be

$$\int_D \omega = \int_D \omega\, dV = \int_D f\, dx_1 \wedge dx_2 \ldots \wedge dx_n.$$

To extend our definition from domains of integration to arbitrary open sets, we need to first restrict the kind of differential n-form that we are working with.

Definition 3.3. Let $U \subset \mathbb{R}^n$ be open and let $\mathbf{v}$ be a vector field on U. Its *support* is the set

$$\operatorname{supp}(\mathbf{v}) = \overline{\{q \in U \mid \mathbf{v}(q) \neq 0\}}.$$

We say $\mathbf{v}$ is *compactly supported* if $\operatorname{supp}(\mathbf{v})$ is compact.

Definition 3.4. Let $U \subset \mathbb{R}^n$ be open, and suppose ω is a compactly supported n-form on U. Then

$$\int_U \omega = \int_D \omega$$

for any domain of integration D such that $\operatorname{supp}(\omega) \subseteq D$.

Essentially, we cover our open set as closely as possible with a domain of integration, and then integrate over the domain.

To extend our integral to arbitrary manifolds, we need to consider the concept of orientations.

Definition 3.5. If F is a smooth map that restricts to an orientation preserving or reversing isomorphism, let $\operatorname{orient}(F)$ be -1 if F is orientation preserving and 1 if F is orientation reversing.

Proposition 3.6. *Let D and E be open domains of integration in $\mathbb{R}^n$. If $G : \overline{D} \to \overline{E}$ is a smooth map that restricts to an orientation preserving or reversing diffeomorphism from D to E and ω is an n-form on $\overline{E}$, then*

$$\int_D G^*\omega = \operatorname{orient}(G) \int_E \omega.$$

Proof. We consider coordinates $(y_1, \ldots, y_n)$ over E and $(x_1, \ldots, x_n)$ over D. Considering ω as $f\, dx_1 \wedge \ldots \wedge dx_n$, we find

$$\int_E \omega = \int_E f\, dV = \int_D (f \circ G)|\det DG| dV,$$

where the proof of the validity of the last manipulation is not proven here but is given in [Lee12]. We can then simplify further to find

$$\begin{aligned}
\int_D (f \circ G)|\det DG| dV &= \int_D (f \circ G) \operatorname{orient}(G)(\det DG) dV \\
&= \int_D (f \circ G) \operatorname{orient}(G)(\det DG)\, dx_1 \wedge \ldots \wedge dx_n \\
&= \int_D G^*\omega.
\end{aligned}$$

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Proposition 3.7. *Let U and V be open subsets of $\mathbb{R}^n$. If $G : U \to V$ is an orientation preserving or reversing diffeomorphism and ω is a compactly supported n-form on V, then*

$$\int_V \omega = \text{orient}(G) \int_U G^*\omega. \text{ [Lee12]}$$

Proof. Consider an open domain of integration E satisfying $\omega \subseteq E \subseteq \overline{E} \subseteq V$. Then $G^{-1}(E) \subseteq U$ is an open domain of integration satisfying $G^{-1}(E) \supseteq \operatorname{supp} G^*\omega$, so by Proposition 3.6, we are done. ∎

Definition 3.8. Let M be a smooth, oriented n-manifold, and ω be a compactly supported differential n-form in the domain of an oriented smooth coordinate chart (U, φ).

We define the integral of ω over M to be

$$\int_M \omega = \int_{\varphi(U)} (\varphi^{-1})^*\omega. \text{ [GH18]}$$

Essentially, we can integrate independently over the pullback of the inverse of each chart at every point in the image of the chart, which allows us to define a concept of the 'oriented' integral on a manifold. This allows integration to be independent of the choice of charts used.

4. Conclusion

Given this definition, we can expand multivariable calculus to apply over manifolds of any dimension. In particular, the generalized Stokes' theorem also holds with this definition. Last, our definition of integration agrees with all definitions from multivariable calculus, which makes it convenient to use generally.

References

[GH18] Victor Guillemin and Peter J. Haine. *Differential Forms.* 2018.
[Lee12] John M. Lee. *Introduction to Smooth Manifolds.* Springer New York, NY, New York, 2012.