LIE GROUPS AND LIE ALGEBRAS

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Contents

1. Introduction

I will assume the reader knows group theory and some linear algebra.

I will then provide some motivation as to why Lie groups and Lie algebras are interesting. Lie groups provide a way to express the concept of a continuous family of symmetries for geometric objects. By differentiating the Lie group action, we get a Lie algebra action, which is a linearization of the group action. As a linear object, a Lie algebra is often easier to work with than working directly with the corresponding Lie group.

Moreover, the general theory of Lie groups and algebras leads to a rich assortment of important explicit examples of geometric objects.

2. Basic Definitions

We can think of a Lie group as a set which has both the structure of a smooth manifold and a group, and that both of these structures play well together

Definition 1. *A set G is called a **Lie Group** iff G is a smooth manifold and G is a group such that the group operations of multiplication and inversion are smooth on the manifold.*

Definition 2. *The tangent space to a Lie Group at the identity element is called the **Lie algebra** of that Lie group. We let $\mathfrak{g}$ stand for the Lie algebra of G. A Lie algebra comes with a bilinear map (called "bracket")*

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which satisfies the following axioms

(1) *Skew-symmetry :*

$$\forall x, y \in \mathfrak{g}, [x, y] = -[y, x]$$

or equivalently

$$\forall x \in \mathfrak{g}, [x, x] = 0$$

(2) *Jacobi Identity :*

$$\forall x, y, z \in \mathfrak{g}, [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

We may also use the notation L to refer to a Lie algebra sometimes.

You can think of the Jacobi identity as a measure of how non-associative the space is. For some examples of such a map with these properties, consider the commutator $[x, y] = xy - yx$. From linear algebra, you may verify that the cross product on $\mathbb{C}^3$ satisfies these axioms, i.e $x \times x = 0$ and $x \times (y \times z) + z \times (x \times y) + y \times (z \times x) = 0$

Definition 3. *If we have a Lie algebra such that*

$$\forall x, y \qquad [x, y] = [y, x]$$

*then we say L is **abelian**. This is not very interesting because it implies that $[x, y] = 0$*

3. Example 1: Circle Group

Let's present a simple example of a Lie Group.

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

where multiplication is complex multiplication and inversion is complex inversion. See figure 1.

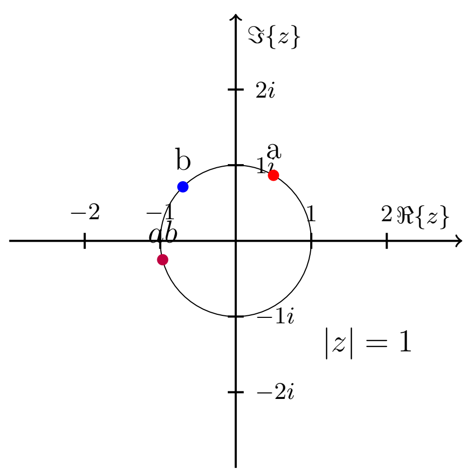

Figure 1. The Circle Group is a Lie Group

Now let's look at the tangent space of the $\mathbf{T}$ at the identity (which is $z = 1$), which is a lie algebra. See figure 2. The tangent space is

1-dimensional, so due to the bilinearity condition, the bracket operator is actually trivial, i.e. $\forall x, y \in \mathfrak{g}, [x, y] = 0$.

It turns out that every 1-dimensional Lie algebra is trivial because all elements of a 1-dimensional vector space can be written as a multiple of the single basis vector, so if we call that basis vector x, then $[\alpha x, \beta x] = \alpha\beta[x, x] = 0$.

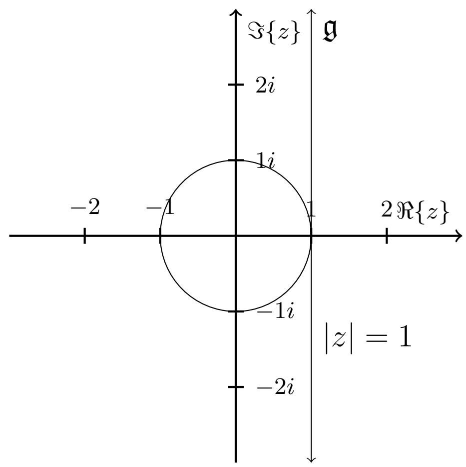

FIGURE 2. Tangent Space of $\mathbb{T}$ at $z = 1$

4. $SL_2(\mathbb{R})$

Now let us look at another Lie Group which has a more interesting Lie Algebra.

Definition 4.

$$SL_2 := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \forall a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$

Where multiplication is matrix multiplciation and inversion is matrix inversion.

It is harder to draw SL_2 because it is 3-dimensional. Once again, let's look at the tangent space at the identity $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. I will use an alternate definition of the tangent space. Given a function $\gamma : \mathbb{R} \to SL_2$ where $\gamma(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we say that $\gamma'(0)$ is an element of the tangent space at I. For the sake of notation, let

$$\gamma(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix} \qquad \gamma'(t) = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

By the definition of SL_2, we have that

$$ad - bc = 1$$

If we take the deriviative of both sides with respect to t at time $t = 0$, we get.

$$\begin{aligned} &\frac{d}{dt}(ad - bc) = \frac{d}{dt}(1) \\ \iff &a'(0)d(0) + a(0)d'(0) - b'(0)c(0) - b(0)c'(0) = 0 \\ \iff &x1 + 1w - y0 - z0 = 0 \\ \iff &x + w = 0 \end{aligned}$$

This means we have found the Lie algebra, which we usually call sl_2. We can write sl_2 as

$$sl_2 = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} : x + w = 0 \right\} = \left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} \right\}$$

From linear algebra, we can see that sl_2 is spanned by 3 matricies (because it is 3 dimensional)

$$sl_2 = span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

The bracket on sl_2 is defined by the commutator $[X, Y] = XY - YX$.

If we label the basis of the Lie algebra as

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then we may calculate that

$$[h, e] = 2e \qquad [h, f] = 2f \qquad [e, f] = h$$

Note that we do not need to calculate any other terms because $[x, y] = -[y, x]$. Something interesting is that h appears to play an "eigenvalue" role here, because when bracketing h on the left, the other parameter is scaled.

5. Representation Theory of Lie Algebras

In this section, we will define a representation of a Lie Algebra and why we care about representations.

Representation give us a way of classifying and categorizing certain Lie Algebras.

Definition 5. *A* ***Lie Algebra representation*** *is a Lie Algebra homomorphism*

$$\mu : \mathfrak{g} \to \mathfrak{gl}(V)$$

By which we mean that

$$\forall x, y \in \mathfrak{g} \qquad \mu([x,y]) = [\mu(x), \mu(y)]$$

Definition 6. *We say a representation μ is* ***faithful*** *iff μ is injective.*

Example 7. *A simple example of a Lie algebra representation is the* ***adjoint representation*** *given by*

$$ad_x : \mathfrak{g} \to \mathfrak{gl}(V)$$
$$ad_x(y) = [x, y]$$

We can prove that the adjoint representation is indeed a representation

$$\begin{aligned}
&[ad_x, ad_y](z) \\
&\qquad = ad_x ad_y(z) - ad_y ad_x(z) \\
&\qquad = ad_x([y,z]) - ad_y([x,z]) \\
&\qquad = [x,[y,z]] - [y,[x,z]] \\
&\qquad = [x,[y,z]] + [[x,z],y] \\
&\qquad = [[x,y],z] \\
&\qquad = ad_{[x,y]}(z)
\end{aligned}$$

To show something cool, we will work towards Ado's Theorem, which is an interesting theorem on Lie algebras.

Theorem 8. ***Ado's Theorem*** *: Every finite dimensional Lie algebra has a faithful finite-dimensional representation.*

Consider the adjoint representation ad_x. Unfortunately, ad_x is not a faithful representation, since $\ker(ad_x) = Z(\mathfrak{g})$. Still, this is a good first guess.

In the case that L is abelian, the center $Z(L) = L$ is easier to deal with. Consider the vector space $L \times K$ (where K is a field of characteristic 0). Consider the representation

$$\forall l \in L, l' \in L, t \in K \qquad \psi(l)(l', t) = (tl, 0)$$

ψ is a representation because

$$\psi([l,l'])(l',t) = (0,0) = \psi(l)(tl'',0) - \psi(l'')(tl,0) = (\psi(l)\psi(l'') - \psi(l'')\psi(l))(l',t)$$

And it may be checked that ψ is a faithful representation.

The gist of Ado's theorem is that we may use smaller representations such as ψ to build up progressively larger representations of larger subalgebras of L which remain faithful.

Taking the direct sum of such a representation on L with the adjoint representation yeilds a faithful representation of L.

Unfortunately, this is not a full proof, because the full proof is too long.

6. Engel's Theorem

Now, we will show an interesting theorem related to the representations of Lie Algebras, known as Engel's Theorem.

The definitions here will mirror those in group theory, if you have familiarity.

Definition 9. *We say I is an **ideal** of a Lie Algebra $\mathfrak{g}$ iff*

$$\forall x \in \mathfrak{g} \forall y \in I \qquad [x, y] \in I$$

Essentially, an ideal "absorbs" elements outside the ideal.

Example 10. 0 *is an ideal of every Lie algebra*

Example 11. *The **center***

$$Z(\mathfrak{g}) := \{z : \forall z \in \mathfrak{g}, \forall x \in \mathfrak{g}, [x, z] = 0\}$$

is an ideal

Notation 12. *In an (abuse of) notation, if I and J are both ideals of $\mathfrak{g}$, then*

$$[I, J] := \{[x, y] : \forall x \in I, \forall y \in J\}$$

and $[I, J]$ is also an ideal of $\mathfrak{g}$

Definition 13. *We define the **central series** as*

$$\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]], \ldots$$

*We say $\mathfrak{g}$ is **nilpotent** if some (and hence all subsequent) elements of the central series are 0.*

Alternitively, we may write that

$$\mathfrak{g}^1 = \mathfrak{g} \qquad \mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n]$$

And then $\mathfrak{g}$ is nilpotent iff $\mathfrak{g}^n = 0$ for some n.

Definition 14. *We say $x \in \mathfrak{g}$ is called **ad-nipotent** iff ad_x is a nilpotent endomorphism, i.e.*

$$\exists k \in \mathbb{N} \qquad (ad_x)^k = 0$$

Theorem 15.
(1) *If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then so it $\mathfrak{g}$*
(2) ***reverse Engel**: If $\mathfrak{g}$ is nilpotent, then all the elements of $\mathfrak{g}$ are ad-nilpotent.*

Proof: We will give a proof of this.

(1) Say $\mathfrak{g} \subset Z(\mathfrak{g})$, then $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n] = [\mathfrak{g}, Z(\mathfrak{g})] = 0$.

(2) The last nonzero term of the descending central series is central.

□

The converse of 15 is also, surprisingly, true

Theorem 16. ***Engels Theorem***: *If all the elements of $\mathfrak{g}$ are ad-nilpotent, then $\mathfrak{g}$ is nilpotent.*

Proof: We give a sketch of the proof of Engel's Theorem.

We are given a Lie algebra of $\mathfrak{g}$ all of whose elements are ad-nilpotent; therefore , the algebra $ad(L) \subseteq \mathfrak{gl}(\mathfrak{g})$ (we can assume $\mathfrak{g} \neq 0$) and it is an exercise to the reader to show that there exists $x \neq 0$ in $\mathfrak{g}$ such that $[\mathfrak{g}, x] = 0$, or in other words $Z(\mathfrak{g}) \neq 0$. Now $\mathfrak{g}/Z(\mathfrak{g})$ consists of ad-nilpotent elements and has smaller dimesnion than $\mathfrak{g}$. Using induction on $dim(\mathfrak{g})$, we find that $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. Theorem 15 implies that $\mathfrak{g}$ itself is nilpotent. □

7. Example 3 : Heisenberg Lie Algebra

We will now look at a Heisenberg Algebra, which is another example of Lie Algebras. The Heisenberg algebra is mostly studied in quantum mechanics, but we will look at it through the lens of Lie groups and Lie algebras.

Definition 17. *The* ***Heisenberg Lie Group*** *is*

$$H = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : \forall a, b, , c \in \mathbb{R} \right\}$$

The heisenberg group is a simply connected Lie Group.

We may take the derivative at I_3 to get

Theorem 18. *The* ***Heisenberg Lie Algebra*** *is a 3-dimensional Lie algebra spanned by three matrices:*

$$p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem 19. *Using the commutator as the Lie Bracket, we may see that*

$$[p, q] = z \qquad [p, z] = 0 \qquad [q, z] = 0$$

From this, we call z the central element, because it is an element of the center. Using definition 13, we may see that the Heisenberg Lie algebra is nilpotent.

This ends our discussion of the Heisenberg Lie algebra.

8. Conclusion

We have looked at Lie groups, and how they relate to manifolds. We have seen how Lie algebras relate to Lie groups. We have also looked at the representation of Lie algebras. Lie algebras have interesting applications in physics, particularly in quantum physics, where the Heisenberg Lie algebra is studied.