The Halfway Method of Sphere Eversion: A Survey

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1 Introduction

Sphere eversion is the process of turning a sphere inside-out without making any pinches, creases, or tears. Let us begin by giving a brief history. It is not at all clear that you can evert the sphere, and the result was first obtained incidentally (one might say accidentally) as a corollary to a key result of Smale [10] which stated that all immersions of the sphere into $\mathbb{R}^3$ are regular homotopy equivalent. However, while this proof could be unwound in theory to provide a construction, the proof was too complex to be unwound in practice. Thus, the problem of giving a simple construction remained open and interesting. The first such direct construction was given by Arnold Shapiro [3] using Boy's surface, followed by Tony Phillips [9]. Since then, many proofs and constructions of sphere eversion have been given, and several constructions minimizing Willmore energy (which can be intuitively thought of as how much a surface deviates from a sphere) have been found. In this paper, we will begin by introducing the formal definitions of sphere eversion, and then consider several reasons why the circle ($\mathbb{S}^1$) cannot be everted. Then, we will discuss the halfway method for sphere eversion, beginning with a survey of Willmore energy and then an in-depth discussion of two its critical points (specifically Morin's surface and Boy's surface). Finally, we will discuss a key way to generalize these two surfaces via the tobacco pouch constructions to an infinite class of viable halfway surfaces.

Now, let us formally define what it means to evert a sphere.

Definition 1.1. An *eversion* of $\mathbb{S}^n$ is a homoptopy $f : \mathbb{S}^n \times [0,1] \to \mathbb{R}^{n+1}$ without creasing such that $f(0) = \mathbb{S}^n$ and $f(1) = -\mathbb{S}^n$. More formally, a crease can be defined as a point $x \in S^n$ and a time t such that the curvature of $f(x,t)$ along some line is not defined or is unbounded.

For $n = 1$, this is analagous to shrinking a loop to a point.

2 Circle Eversion

While sphere eversion is, surprisingly, possible, its one-dimensional counterpart is not.

Theorem 2.1. *It is impossible to evert* $\mathbb{S}^1$.

2.1 Total curvature

Definition 2.2. For a curve parameterized by $t \in [0,1)$, consider the continuous angle function $a(t)$ satisfying $a(0) = 0$ and for sufficiently close x and y, $a(y) - a(x)$ is the difference between the angles of the tangent lines at x and y. We use the condition that the curvature is defined and bounded to ensure that this definition is consistent. Then, let $a_f = \lim_{t \to 1^-} a(t)$. The angle $a_f - a(0)$ is called the *total curvature*.

Lemma 2.3. *The total curvature of a curve is homotopy invariant.*

Proof. Since a homotopy is a continuous deformation, we can consider deforming infinitesimal segments. Thus, consider four arbitrarily close points w, x, y, z on our curve. If we deform the path connecting x and y, the value of $a(z)$ will remain unchanged since $a(z) - a(w)$ is the difference in tangent line angles at w and z. Hence, our deformation maintains the total curvature. □

Armed with this proof, we can now prove Theorem 2.1.

Proof of Theorem 2.1. The circle $\mathbb{S}^1$ when traversed counterclockwise has a total curvature of 2π. However, when its orientation is reversed, it has a total curvature of -2π. Since total curvature is homotopy invariant, we cannot evert the circle. □

2.2 Gauss Map

Definition 2.4. At a point $x \in X$, let $\boldsymbol{n}_x$ be the normal vector. Then, if X is a closed curve in $\mathbb{R}^2$, we can define the map $f : X \to \mathbb{S}^1$ defined by $f(x) = \boldsymbol{n}_x$. Then, the *Gauss number* of the manifold X is the degree of f.

Lemma 2.5. *The Gauss number of a manifold is homotopy invariant.*

Proof. Since a homotopy on X corresponds to a homotopy on f and degree is homotopy-invariant, the degree of f remains unchanged under homotopy. □

We can now prove Theorem 2.1 again.

Proof of Theorem 2.1. The circle $\mathbb{S}^1$ clearly has degree 1, while the everted circle has degree -1. Thus, we cannot evert the circle. □

3 Willmore Energy

The Willmore energy of a surface is the integral of the square of the mean curvature minus the Gaussian curvature.

$$\mathcal{W} = \int_S (H^2 - K) dA$$

There are many other equivalent representations of this quantity given via differential geometry. For example, we have:

$$\mathcal{W} = \int_S H^2 dA 2\pi\chi(S)$$
$$\mathcal{W} = \frac{1}{4}\int_S (k_1 - k_2)^2 dA$$

The final equation given in terms of the principle curvatures is the most intuitive. At a high level, it tells us that the Willmore energy of a surface measures how irregularly curved that surface is, and at an even higher level, the Willmore energy tells us how much a surface deviates from being spherical. Now, the Willmore energy of the sphere becomes 0, since both principle curvatures are the same everywhere.

Since the sphere is a local minima of the Willmore energy, the following approach has been employed to construct a sphere eversion. Start from some surface which has no tendency towards either orientation of the sphere, meaning that it is highly symmetric with respect to flipping orientation. From here, we may apply gradient descent on Willmore energy (known as Willmore flow). This process has a high likelyhood of reaching the sphere as its local minima as opposed to some other arbitrary shape. Since this process is symmetric with respect to orientation, this gives us a valid sphere eversion.

The method described above requires heavy-duty computer simulations and, when applied to Morin's surface, yields the so-called minimax eversion.

In the types of eversions described above, the use of a halfway surface is employed which allows us to recover a complete eversion from a completed Willmore flow. In addition, by Smale's original theorem, all immersions of the sphere are regular homotopy equivalent, so any halfway surface can be transformed into a sphere using gradient descent, as long as we do not get trapped in a local minima. Thus, in the modern context, the only required piece to generate a new eversion is a halfway surface.

Another question connecting sphere eversion and Willmore energy is about the minimum Willmore energy over all eversions. One well-known result concerning Sphere eversion is that any eversion must pass through

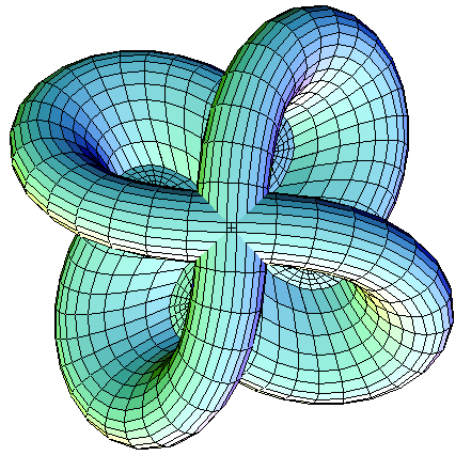

Figure 1: Morin's Surface (Source: Wikipedia)

some stage with a quadruple point, and another well-known result states that any surface with k coinciding points for $k > 1$ must have Willmore energy at least k. Putting these two results together, every sphere eversion must pass through some stage with Willmore energy ≥ 4. However, Morin's surface, which we discuss in the following section, turns out to be a halfway surface with a Willmore energy of 4, confirming that the minimum sphere eversion has energy 4 [7, 5, 4, 8].

4 Morin's Surface

Morin's Surface is a particular immersion of the sphere with high levels of symmetry between the inside and outside of the sphere. Informally, the surface contains four different symmetric bands, where opposite bands are of the same orientation. Thus, if we convert a standard sphere immersion regularly to Morin's surface, we can simply rotate the surface $90°$ and undo the homotopoy to obtain an everted sphere. Hence, Morin's surface is an example of a halfway surface, and since its Willmore energy is 4, it provides a minimal sphere eversion.

Morin's Surface can best be understood via the regular homotopy from a sphere that creates it. To obtain Morin's surface, we start by pushing the North pole of the sphere downwards towards the South pole, giving us a figure that resembles half of a sphere. Also color the outside red and the inside blue for convenience. Then, we take the top portion of the sphere and we create two arms from the top, both of which should be red. From the bottom of the sphere, we take the blue part of the sphere and form two arms which we also arrange at the top. This gives us Morin's surface, and so a half-turn interchanges the inside and the outside. We can then reverse the process to interchange the inside and outside. This process is visualized here.

Morin's Surface can be parameterized in Cartesian coordinates with the following equations in u, v:

$$K = \frac{\cos u}{\sqrt{2} - \sin 2u \sin v}$$
$$x = K\left(2\cos u \cos v + \sqrt{2}\sin u \cos v\right)$$
$$y = K\left(2\cos u \sin v - \sqrt{2}\sin u \sin v\right)$$
$$z = K \cos u.$$

This parameterization is quite long and involves many trignometric functions, but we will see later that it can be extended to other surfaces very naturally.

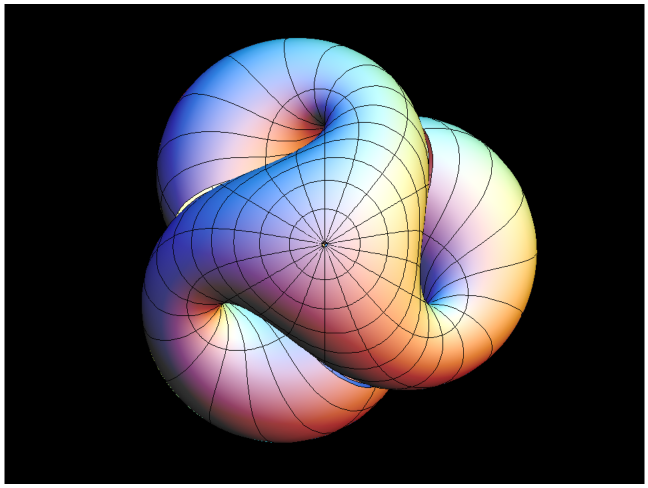

Figure 2: Boy's Surface (Source: Virtual Math Museum)

5 Boy's Surface

Boy's surface is an immersed projective plane, or equivalently, it is an immersion of the sphere where antipodal points have the same location. Boy's surface is of particular interest with respect to Sphere eversion, as shown in Figure 2. This Boy's surface manages to immerse the projective plane via the following parameterization.

Let:

$$g_1 = -\frac{3}{2}\Im\left[\frac{z\left(1-z^4\right)}{z^6+\sqrt{5}z^3-1}\right]$$

$$g_2 = -\frac{3}{2}\Re\left[\frac{z\left(1+z^4\right)}{z^6+\sqrt{5}z^3-1}\right]$$

$$g_3 = \Im\left[\frac{1+z^6}{z^6+\sqrt{5}z^3-1}\right]-\frac{1}{2}$$

for some complex number z in the complex unit disk (i.e. $|z| \leq 1$). Then, if we allow $g = g_1^2 + g_2^2 + g_3^2$, the Cartesian coordinates of the mapped point z are $(x, y, z) = \frac{1}{g}(g_1, g_2, g_3)$.

However, dealing with this explicit parameterization is beyond the scope of this paper, so we will instead focus on describing Boy's surface geometrically. This surface is best understood by considering a regular homotopy from a circle in three dimensions to another circle where antipodal points map to each other. This process is best visualized by this video. At a high level, we consider 3 opposite pairs of equally-sized regions of our circle, and by folding our circle in a way symmetric with respect to these regions, we can obtain the desired shape.

To obtain the immersion from here, we can remove the poles of our sphere, do the above folding to overlap opposite sides of the resulting figure, and then re-extend our three regions to meet at the poles, we can obtain the Boy's surface.

However, the more difficult part of using a Boy's surface to evert the sphere is not describing the surface itself but finding a regular homotopy from a standard sphere immersion to a double covering of the Boy's surface. This can be done using Willmore flow. Since Boy's surface is a saddle point of Willmore energy (due to its tremendous symmetry), it can reach either orientation of the sphere after Willmore flow. In fact, halfway eversions utilizing Boy's surface were the first eversions constructed.

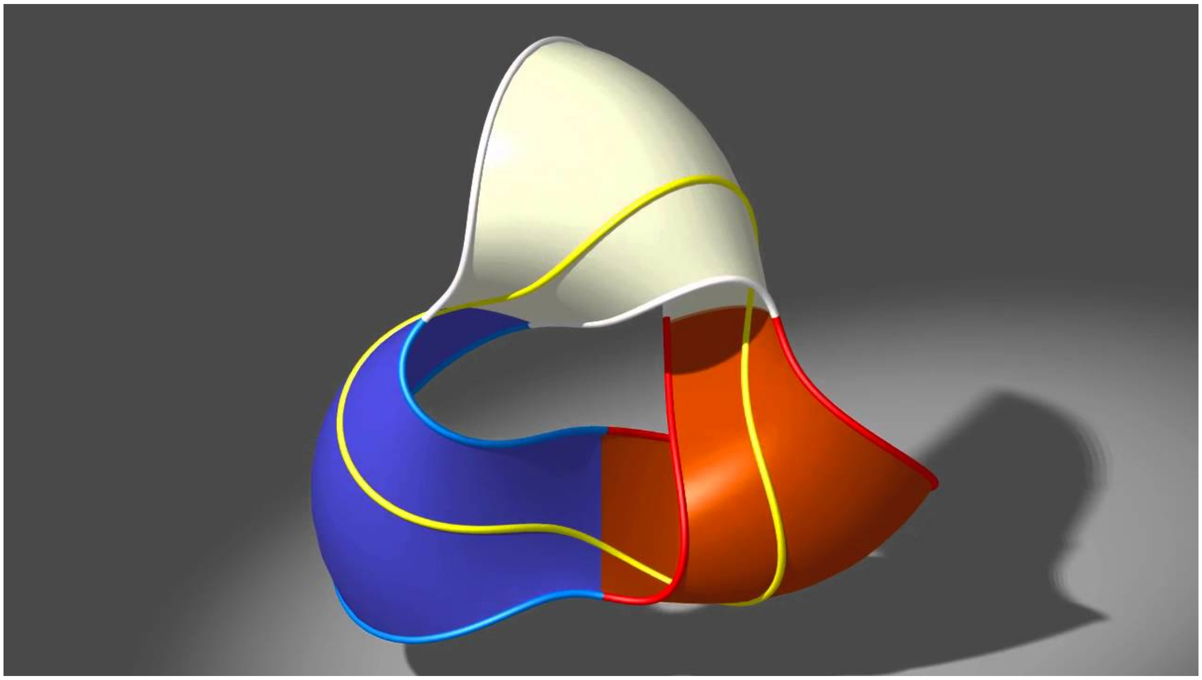

Figure 3: The portion of Boy's Surface corresponding to the equator of a sphere (Source: Jos Leys, YouTube)

6 Tobacco-Pouch Eversions

While Morin's surface may be the energy-minimizing halfway surface, and Boy's surface was used as part of the first ever sphere eversion, these surfaces are only the first two elements of an infinite class of surfaces known as the tobacco-pouch surfaces [2]. Morin's surface is the $n = 2$ case of the tobacco-pouch construction and when rotated by a $\frac{1}{2}$ rotation, we get back the same surface (note that rotating by $\frac{1}{4}$ of a full turn interchanges the inside and outside). Similarly, Boy's surface is the $n = 3$ case of the construction, having three-fold rotational symmetry.

The tobacco pouch construction generalizes these two surfaces to higher numbers. For even n, it gives a surface resembling Morin's surface except with $2n$ arms instead of 4.

For odd n, the tobacco pouch construction generalizes Boy's surface, and instead of the inside and the outside being a $\frac{1}{2n}$ full rotation off of each other, the inside and the outside double cover the surface, giving

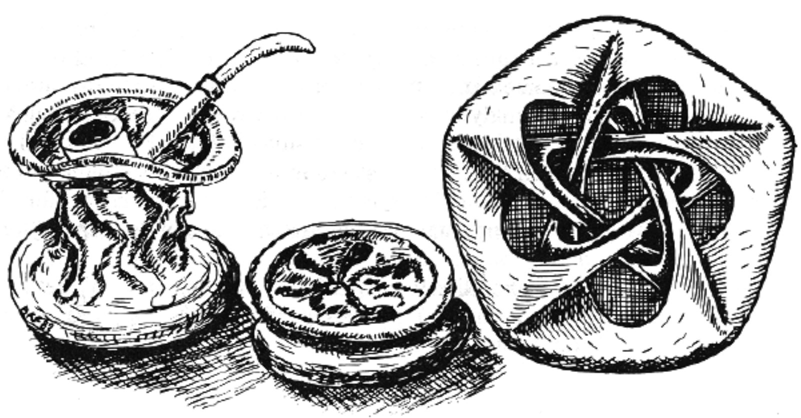

Figure 4: The odd case of the tobacco pouch construction [2]

an immersion of the real projective plane.

As we have seen before, both of these types of halfway models are extremely powerful since they can easily be converted into sphere eversions via Willmore flow.

Now, it may seem like the even and odd case of the tobacco pouch construction have nothing to do with each other, but both cases can be parameterized in the same way. If we let n be the input to the construction, u, v be parameters, and k be some variable controlling the particular embedding, the n^{th} surface can be parameterized as follows:

$$K = \frac{\cos u}{\sqrt{2} - k \sin 2u \sin nv}$$
$$x = K\left(\frac{2}{n-1}\cos u \cos\left((n-1)v\right) + \sqrt{2}\sin u \cos v\right)$$
$$y = K\left(\frac{2}{n-1}\cos u \sin\left((n-1)v\right) - \sqrt{2}\sin u \sin v\right)$$
$$z = K\cos u.$$

If we plug in $n = 2, k = 1$, this yields the standard embedding of Morin's surface, and plugging in $n = 3, k = 1$ gives Boy's surface.

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