Hyperbolic Geometry

Xinyang(Thalia) Qian

August 2025

Abstract

This paper gives an elementary introduction to the geometry of the hyperbolic plane. Inspired by the Euclidean Axiom and doing the alternation of the Parallel Postulate, hyperbolic geometry comes to the stage as an example of Non-Euclidean geometry. In Section §1, this paper is going to discuss the Hyperbolic Axiom. In Section §2, this paper covers five models of a hyperbolic plane and the mappings between them. In Sections §$3-5$, this paper discusses essential properties of Hyperbolic Geometry spanning from hyperbolic plane, hyperbolic arclengths and areas, to hyperbolic isomotries. In Sections §$6-9$, this paper discusses further topics, including Mostow's theorem, Kleinian groups, algebraic convergence, and the Techm*ü*ller space.

Contents

There are three kinds of geometry: elliptic geometry (or spherical geometry), Euclidean of parabolic geometry, and hyperbolic or Lobachevskiian geometry. The underlying spaces of these three geometries are Riemannian manifolds of constant sectional curvature +1, 0, -1, respectively. An example of a Riemannian manifold with constant curvature -1 that indicates some of the properties of parabolic geometry is a pseudosphere.

1 An Example to understand Hyperbolic Geometry

A sphere in Euclidean space with radius r has constant curvature $\frac{1}{r^2}$. Thus, hyperbolic space should be a sphere of radius i. To give this a reasonable interpretation, we use an indefinite metric $dx^2 = dx_1^2 + ... + dx_n^2 - dx_{n+1}^2$ in $\mathbb{R}^{n+1}$. The sphere of radius i about the origin in this metric is the hyperboloid

$$x_1^2 + ... + x_n^2 - x_{n+1}^2 = -1. \tag{1}$$

The metric dx^2 restricted to the hyperboloid is positive definite, and it is not hard to check that it has constant curvature -1.

The Theoroma Egregium states that any point of a surface of constant Gaussian curvature is contained in a patch that is isometric to an open subset of a plane, a sphere, and a pseudosphere. Hence, understanding the pseudosphere helps us understand any space with constant curvature -1, hence helping us understand the hyperbolic geometry.

Definition 1.1 (Pseudosphere)**.** The pseudosphere is the constant negative-Gaussian curvature surface of revolution generated by a tractrix about its asymptote. Its parametrization is

$$\gamma(u, v) = (u \cos v, u \sin v, u - \tanh u), \tag{2}$$

$u \geq 0$, $v \in [0, 2\pi]$.

One of the important properties of the pseudosphere is that its profile curve (the tractrix) is determined up to translation.

Proposition 1. *The tractrix is determined up to translation. This can be proved by the two equivalent statements as below:*

- *(i) The tangent lines of the tractrix meet the x-axis a unit distance from a point of tangency.*

- *(ii) Let γ be the tractrix in the xz-plane, given by the $z = \sqrt{1-x^2} - \cosh^{-1}\frac{1}{x}$, and let $\mathbf{p}$ be a point on γ. Suppose the tangent line to γ at $\mathbf{p}$ intersects the z-axis at $\mathbf{q}$. The distance from $\mathbf{p}$ to $\mathbf{q}$ is always 1.*

Proof. Let

$$x = \frac{1}{\cosh t}, z = \tanh t - t, \tag{3}$$

then the tangent vector at $\mathbf{p}$ is given by

$$\frac{dx}{dt} = \frac{d}{dt}\frac{1}{\cosh t} = -\frac{1}{\cosh t}\tanh t, \tag{4}$$

$$\frac{dz}{dt} = \frac{d}{dt}(\tanh t - t) = \frac{1}{\cosh^2 t} - 1, \tag{5}$$

$$\tag{6}$$

, hence

$$\frac{dz}{dx} = \frac{1-\cosh^2 t}{-\cosh t \tanh t} = \frac{\sinh^2 t}{\sinh t} = \sinh t. \tag{7}$$

Hence the slope at $\gamma(x,z) = \gamma(x(t), z(t)) = (\frac{1}{\cosh t}, \tanh t - t)$ is $\sinh t$. The equation of the tangent line is

$$z - z(t) = \sinh t(x - x(t)). \tag{8}$$

Let $\mathbf{q} = (0, z_q)$, then we have

$$z_q - \tanh t + t = \sinh t(0 - \frac{1}{\cosh t}) = -\frac{\sinh t}{\cosh t} = -\tanh t. \tag{9}$$

Hence we have $z_q = t$, hence $\mathbf{q} = (0, -t)$. Therefore, the distance from $\mathbf{p}$ to $\mathbf{q}$ is

$$\sqrt{(\frac{1}{\cosh t})^2 + (\tanh t - t - (-t))^2} = \sqrt{\frac{1}{\cosh^2 t} + \tanh^2 t} = \sqrt{1} = 1. \tag{10}$$

Hence, the distance from $\mathbf{p}$ to $\mathbf{q}$ is always 1. □

Another important property about the pseudosphere is that it is not complete because it has an edge——beyond which it cannot be extended. This property is proved by Hilbert.

Theorem 1 (Hilbert's Theorem)**.** *There exists no isometric immersion of a complete surface with constant negative Gaussian curvature in $\mathbb{R}^3$.*

Hilbert's Theorem describes how the intrinsic geometry of the hyperbolic plane cannot be reconciled with the external structure of $\mathbb{R}^3$.

2 The Discovery of Non-Euclidean Geometry

J*á*nos Bolyai(1802-1860) announced privately his discoveries in non-Euclidean geometry, as a 26-page appendix to a book by his father Farkas surveying attempts to prove Euclid V (The Tentamen, 1831). Farkas sent a copy to his friend, the German mathematician Carl Friedrich Gauss(1777-1855). Farkas Bolyai had become close friends with Gauss 35 years earlier, when they were both students in G*ö*ttingen. After Farkas returned to Hungary, they remained an intimate correspondence, and when Farkas sent Gauss his own attempt to prove the parallel postulate, Gauss tactfully pointed out the fatal flaw.

However, Gauss's conservative attitude towards publicizing J*á*nos's achievements made J*á*nos fall into depression and never again published his research. His father did not understand J*á*nos's discovery and subsequently published another clever attempt to prove Euclidean V.

In 1817, Gauss wrote to W. Olbers: " I am becoming more and more convinced that the necessity of our [Euclidean] Geometry cannot be proved." In 1824, Gauss answered F.A Traurinus, who had attempted to investigate the theory of parallels, that it is possible that the sum of the angles can be less than 180°. Such an assumption led to a curious geometry.

Another actor in the historical drama came along to steal the limelight from both J. Bolyai and Gauss: the Russian mathematician Nikolai Ivanovich Lobachevsky (1792-1856). He was the the first to actually publish an account of non-Euclidean geometry, in 1829. In an 1846 letter to Schumacher, Gauss reiterated his own priority in developing non-Euclidean geometry but conceded that "Lobachevsky carried out the task in a masterly fashion and in a truly geometric spirit." At Gauss' recommendation, Lobachevsky was elected to the G*ö*ttingen Scientific Society. (Why didn't Gauss recommend J*á*nos Bolyai?)

As a saying of Gauss put, "we must not put geometry on par with arithmetic that exists purely a priori but rather with mechanics." The great French mathematicians J. L. Lagrange (1736-1813) and J. B. Fourier (1768-1830) tried to derive the parallel postulate from the law of lever in statics.

It is amazing how similar are the approaches of J. Bolyai and Lobachevsky. Both attacked plane geometry via the "horosphere" in three-space (it is the limit of an expanding sphere when its radius tends to infinity). Both showed the geometry on a horosphere, where "lines" are interpreted as "horocycles" (limits of circles), is Euclidean. Both showed that Euclidean spherical trigonometry is valid in neutral geometry and both constructed a mapping from the sphere to the non-Euclidean plane to derive the formulas of non-Euclidean trigonometry. Both had a constant in their formulas that they could not explain; the later work of Riemann showed it to be the curvature of the non-Euclidean plane.

Subsequently, some of the best mathematicians (Beltrami, Klein, Ponicar*é*, and Riemann) took up the subject, extending it, clarifying it, and applying it to other branches of mathematics, notably complex function theory.

In 1868 the Italian mathematician Beltrami settled once and for all the question of a proof for the parallel postulate. He did this by exhibiting a Euclidean model of non-Euclidean geometry.

Bernard Riemann, who was a student of Gauss, had the most profound insight into the geometry. In 1854, he built upon Gauss' discovery of intrinsic geometry on a surface in Euclidean three-space. Riemann invented the concept of an abstract geometrical surface that need not be embeddable in Euclidean three-space yet on which the "lines" can be interpreted as geodesics and the intrinsic curvature of the surface can be precisely defined.

Interestingly, a direct relationship between the spectral theory of relativity and hyperbolic geometry was discovered by the physicist Arnold Sommerfield in 1909 and elucidated by the geometer

Vladimir Varičak in 1912. A model of hyperbolic plane geometry is a sphere of imaginary radius with antipodal points identified in the three-dimensional space-time of special relativity.

3 Hyperbolic Axiom

Hyperbolic geometry is the geometry one can get by assuming all the axioms of neutral geometry and replacing Hilbert's parallel postulate by its negation, which we shall call the hyperbolic axiom. The following axioms are the set-ups of the Hyperbolic Geometry.

Theorem 2 (Euclid's Postulates)**.** *(i) A straight line segment can be drawn joining any two points,*

(ii) Any straight line segment can be extended indefinitely in a straight line,

(iii) Given any straight lines segment, a circle can be drawn having the segment as radius and one endpoint as center,

(iv) ll Right Angles are congruent,

(v) If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two Right Angles, then the two lines inevitably must intersect each other on that side if extended far enough.

These five axioms are the building blocks of the Euclidean Geometry.

Theorem 3 (Parallel Postulate)**.** *Given any straight line and a point not on it, there "exists one and only one straight line which passes" through that point and never intersects the first line, no matter how far they are extended.*

This statement is equivalent to the fifth of Euclid's postulates.

Theorem 4 (Hyperbolic Axiom)**.** *In hyperbolic geometry there exist a line l and a point P not on l such that at least two distinct lines parallel to l pass through P.*

This is a negation of the Parallel Postulate. It states that for some line l and some point P not on l, uniqueness of parallels fails to hold.

Lemma 5. *Rectangles do not exist.*

Proof. Suppose we have a rectangle $ABCD$, then we have $AB \parallel CD$, $AD \parallel BC$. Hence, for any side of the rectangle, if we take an arbitrary point on its opposite side, we will have a line passing through this point that is parallel to the side we have chosen (Namely, take another point on its opposite side different from what we have chosen, and take the segment determined by these two points). This follows that the parallel postulate holds. □

This lemma is equivalent to the statement as follows: the existence of rectangles implies Hilbert's parallel postulate, the negation of the hyperbolic axiom. Using the lemma, we can establish a universal version of Hyperbolic Theorem.

Theorem 6 (Universal Hyperbolic Theorem)**.** *In hyperbolic geometry, for every line l and every point P not on l there pass through P at least two parallels to l.*

Proof. Drop perpendicular $\overline{PQ}$ to l and erect line m through P perpendicular to $\overline{PQ}$. Let R be another point on l, erect perpendicular t to l through R, and drop perpendicular $\overline{PS}$ to t. Now $\overline{PS}$ is parallel to l, since they are both perpendicular to t. We claim that m and $\overline{PS}$ are distinct lines. Assume on the contrary that S lies on m. Then $\square PQRS$ is a rectangle. This contradicts with that lemma that rectangles do not exist. □ This theorem implies that uniqueness of parallels fails for all l and P.

Corollary 6.1. *In hyperbolic geometry, for every line l and every point P not on l, there are infinitely many parallels to l through P.*

Proof. Just vary the point R in the above proof. □

3.1 Angel Sums

Theorem 7. *In hyperbolic geometry, all angles have angle sum < 180°.*

Corollary 7.1. *In hyperbolic geometry all convex quadrilaterals have angle sum less than 360°.*

Proof. Given any quadrilateral $\square ABCD$. Take diagonal AC and consider the triangles $\triangle ABC$ and $\triangle ACD$; by the theorem, these triangles have the angle sum 180°. The assumption that $\square ABCD$ is convex implies that $\overline{AC}$ is between $\overline{AB}$ and $\overline{AD}$ and that $\overline{CA}$ is between $\overline{CB}$ and $\overline{CD}$, so that the angle sum of the convex quadrilateral is the sum of that of two triangles, hence the angle sum of $\square ABCD$ is less than 360°. □

3.2 Similar Triangles

Theorem 8. *In hyperbolic geometry if two triangles are similar, they are congruent.*

Proof. Assume on the contrary that there exist triangles $\triangle ABC$ and $\triangle A'B'C'$ which are similar but are not congruent. Then no corresponding sides are congruent; otherwise the triangles would be congruent (ASA). Consider the triples (AB, AC, BC) and ($A'B'$, $A'C'$, $B'C'$) of sides of these triangles. One of these triples must contain at least two segments that are larger than the two corresponding segments of the other triple, e.g., $AB > A'B'$ and $AC > A'C'$. Then there exists B'' on AB and C'' on AC such that $AB'' \cong A'B'$ and $AC'' \cong A'C'$. By SAS, $\triangle A'B'C' \cong \triangle AB''C''$. Hence, corresponding angles are congruent: $\angle AB''C'' = \angle B'$, $\angle AC''B'' = \angle C'$. This implies that $\overline{BC} \parallel \overline{B''C''}$, so that quadrilateral $\square BB''C''C$ is convex. Also, $\angle B + \angle BB''C'' = 180° = \angle C + \angle BC''B''$. It follows that quadrilateral $\square BB''C''C$ has angle sum 360°. This contradicts to the corollary above. □

3.3 Parallels that Admit a Common Perpendicular

Theorem 9. *In hyperbolic geometry if l and l'' are any distinct parallel lines, then any set of points on l equidistant from l'' has at most two points in it.*

Proof. Assume on the contrary there is a set of three points A, B, and C on l equidistant from l'. Then quadrilaterals $\square A'B'BA$, $\square A'C'CA$, and $\square B'C'CB$ are Saccheri quadrilaterals (the base angles are right angles and the sides are congruent). Through known conclusions, we have $\angle A'AB \cong \angle B'BA$, $\angle A'AC \cong \angle C'CA$, and $\angle B'BC \cong \angle C'CB$. By transitivity, it follows that the supplementary angles $\angle B'BA$ and $\angle B'BC$ are congruent to each other; hence, by definition, they are right angles. Therefore, these Saccheri quadrilaterals are all rectangles. But rectangles do not exist in hyperbolic geometry. This contradiction shows that A, B, and C cannot be equidistant from l'. □

Theorem 10. *In hyperbolic geometry if l and l' are parallel lines for which there exists a pair of points A and B on l equidistant from l', then l and l' have a common perpendicular segment that is also the shortest segment between l and l'.*

Proof. Suppose A and B on l are equidistant from l'. Then $\square A'B'BA$ is a Saccheri quadrilateral, where A' and B' are the feet on l' of the perpendicular from A and B. Let M be the midpoint of $\overline{AB}$ and M' be the midpoint of $\overline{A'B'}$. The theorem will follow the next lemma. □

Theorem 11. *In hyperbolic geometry if lines l and l' have a common perpendicular segment MM', then they are parallel and MM' is unique. Moreover, if A and B are any points on l such that M is a midpoint of segment AB, then A and B are equidistant from l'.*

Proof. We know that $\angle A$ and $\angle B$ are congruent. Hence, $\triangle A'AM \cong \triangle B'BM(SAS)$. Therefore, the corresponding sides $A'M$ and $B'M$ are congruent. This implies $\triangle A'M'M \cong \triangle B'M'M$ are congruent. Since these are supplementary angles, they might be right angles, proving MM' perpendicular to the base $A'B'$. From the two pairs of congruent triangles, we also have $\angle A'MM' \cong \angle B'MM'$ and $\angle A'MA \cong \angle B'MB$. Adding the degrees of these angles, we have $\angle AMM' = \angle BMM'$.

Consider next quadrilateral $\square A'M'MA$. It has three right angles, so it is what we call a Lambert quadrilateral. In hyperbolic geometry the fourth angle must be acute, since rectangles do not exist. Known conclusion gives $AA' > MM'$. The remainder of the proof that MM' is shorter than AA'. MM' is than any other segment between l and l' can also be proved.□

Theorem 12. *If Euclidean geometry is consistent, then so is hyperbolic geometry.*

Theorem 13 (Incidence Axiom 1 (Klein))**.** *Given any two distinct points A and B in the interior of circle γ. There exists a unique open chord l of γ such that A and B both lie on l.*

4 Five Models of Hyperbolic Space

The five models of hyperbolic space are:
(i) H, the Half-space model;
(ii) I, the Interior of the disk model;
(iii) J, the Jemisphere model;
(iv) K, the Klein model;
(v) L, the Loid model.

4.1 The Poincar$\acute{e}$ disk model

4.2 The Southern Hemisphere

The Poincaré disk $D^n \subset \mathbb{R}^n$ is contained in the Poincaré disk $D^{n+1} \subset \mathbb{R}^{n+1}$, as a hyperbolic n-plane in hyperbolic (n+1)-space.

Stereographic projection from the north pole of ∂D^{n+1} sends the Poincaré disk D^n to the southern hemisphere of D^{n+1}.

Thus hyperbolic lines in Poincaré disk go to circles on S^n orthogonal to the equator S^{n+1}.

4.3 The Upper-Half Space Model

To obtain this, rotate the sphere S^n in $\mathbb{R}^{n+1}$ so that the southern hemisphere lies in the half-space $x_n \geq 0$ is $\mathbb{R}^{n+1}$. Now stereographic projection from the top of S^n (which is now on the equator) sends the southern hemisphere to the upper half-space $x_n > 0$ in $\mathbb{R}^{n+1}$. A Hyperbolic

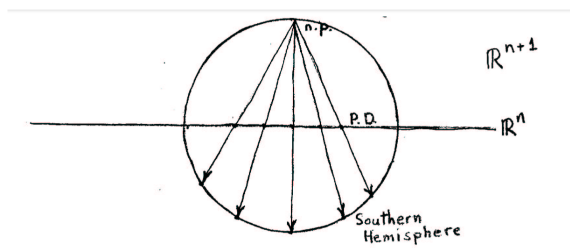

Figure 1: Stereographic Projection onto the Southern Hemisphere

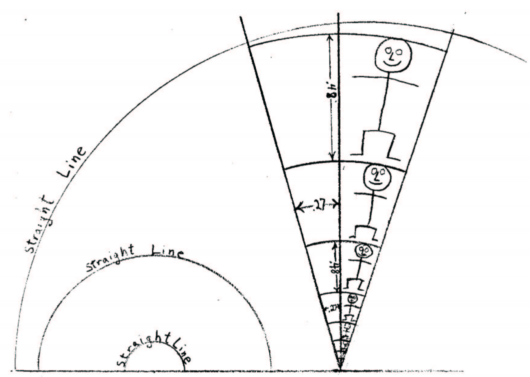

Figure 2: Hyperbolic Lines in the Upper Half-Plane

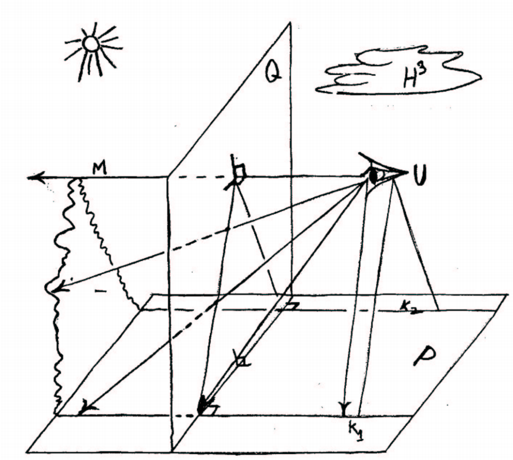

Figure 3: Two lines meet somewhere in the two-dimensional projective model

line, in the upper half-space, is a circle perpendicular to the bounding plane $\mathbb{R}^{n-1} \subset \mathbb{R}^n$. The hyperbolic metric is $ds^2 = \left(\frac{1}{x_n}\right)^2 dx^2$. Thus, the Euclidean image of a hyperbolic object moving toward $\mathbb{R}^{n-1}$ has size precisely proportional to the Euclidean distance from $\mathbb{R}^{n-1}$.

4.4 The projective model

This is obtained by Euclidean orthogonal projection of the southern hemisphere of S^n back to the disk D^n. Hyperbolic lines become Euclidean line segments. This model is useful for understanding incidence in a configuration of lines and planes. Unlike the previous three models, it fails to be conformal, so that angles and shapes are distorted.

It is better to regard this projective model to be contained not in Euclidean space, but in projective space. The projective model is very natural from a point of view inside hyperbolic (n+1)-space: it gives a picture of a hyperplane, H^n, in true perspective. Then, an observer hovering above H^n in H^{n+1}, looking down, sees H^n as the interior of a disk in its visual sphere. As he moves farther up, this visual disk shrinks; as he moves down, it expands; but (unlike in Euclidean space), the visual radius of this disk is always strictly less than $\pi/2$. A line on H^2 appears visually straight.

4.5 Properties of Five Models

Definition 4.1 (Domains of Five Models)**.** The five domains are
H, $(1, x_2, ..., x_{n+1}), x_{n+1} > 0$;
I, $(x_1, ..., x_n, 0) : x_1^2 + ... + x_n^2 < 1$;
J, $(x_1, ..., x_{n+1}) : x_1^2 + ... + x_n^2 = 1$ and $x_{n+1} > 0$;
K, $(x_1, ..., x_n, 1) : x_1^2 + ... + x_n^2 < 1$;
L, $(x_1, ..., x_{n+1}) : x_1^2 + ... + x_n^2 - x_{n+1}^2 = -1$ and $x_{n+1} > 0$.

Definition 4.2 (Riemannian Metrics of Five Models)**.** The associated Riemannian metrics ds^2 of the five models are
$d^2_{s_H} = \dfrac{d^2_{x_2} + ... + d^2_{x_{n+1}}}{x^2_{n+1}}$;
$d^2_{s_I} = 4\dfrac{d^2_{x_1} + ... + d^2_{x_n}}{(1 - x_1^2 - ... - x_n^2)^2}$;
$d^2_{s_J} = \dfrac{d^2_{x_1} + ... + d^2_{x_{n+1}}}{x^2_{n+1}}$;
$d^2_{s_K} = \dfrac{d^2_{x_1} + ... + d^2_{x_n}}{1 - x_1^2 - ... - x_n^2} + \dfrac{(x_1dx_1 + ... + x_ndx_n}{(1 - x_1^2 - .. - x_n^2)^2}$;
$d^2_{s_L} = dx_1^2 + .. + dx_n^2 - dx^2_{n+1}$.

Proposition 2 (The Isometries among Five Models)**.** *We use J as the central model and describe for each of the others a simple map to or form J:*
(i) $\alpha : J \to H$

$$(x_1, ..., x_{n+1}) \mapsto (1, \frac{2x_1}{x_1 + 1}, ..., \frac{2x_{n+1}}{x_{n+1} + 1}) \tag{11}$$

The map $\alpha : J \to H$ *is central projection from the point (-1, 0, ..., 0).*
(ii) $\beta : J \to I$

$$(x_1, ..., x_{n+1}) \mapsto (\frac{x_1}{x_{n+1} + 1}, ..., \frac{x_n}{x_{n+1} + 1}, 0) \tag{12}$$

The map $\beta : J \to I$ *is central projection from (0,..., 0, -1).*
(iii) $\gamma : K \to J$

$$(x_1, ..., x_{n+1}) \mapsto (x_1, ..., x_n, \sqrt{1 - x_1^2 - ... - x_n^2}) \tag{13}$$

The map $\gamma : K \to J$ *is vertical projection.*
(iv) $\delta : L \to J$

$$(x_1, ..., x_{n+1}) \mapsto (\frac{x_1}{x_{n+1}}, ..., \frac{x_n}{x_{n+1}}, \frac{1}{x_{n+1}}) \tag{14}$$

The map $\delta : L \to J$ *is central projection from (0,0,0,..., -1).*

5 The Hyperbolic Plane

5.1 Points and Lines

Definition 5.1. A hyperbolic geodesic in $\mathbb{H}$ is either a straight vertical half-line, or a half-circle centered on the horizontal line.

Proposition 3 (Properties of Hyperbolic Geodesics)**.** *(1) Through any two points on* $\mathbb{H}$ *there is exactly one geodesics.*

(2) If we fix a point in $\mathbb{H}$, there is exactly one geodesic through that point with any prescribed tangent line.
(3) Fix a geodesic c, and a point p not lying on c. Then there are infinitely many geodesics passing through p and which do not intersect c.

5.2 Angles

Theorem 14. *In a hyperbolic triangle, the sum of the angles is always less than π.*

Definition 5.2. The hyperbolic distance between two points $z, w \in \mathbb{H}$ is

$$\text{dist}(z,w) = ln\frac{|z-\bar{w}|+|z-w|}{|z-\bar{w}|-|z-w|}$$

The hyperbolic circle with center (x, y) and radius r is exactly the ordinary Euclidean circle with center $(x, \cosh(r)y)$ and radius $\sinh(r)y$.

6 Hyperbolic Arclengths and Areas

6.1 Lengths and Distances

Definition 6.1. At a point (x, y) in the hyperbolic plane, the infinitesimal length element is

$$\frac{\sqrt{dx^2+dy^2}}{y}.$$

Theorem 15. *For any two points z and w,*

$$\text{dist}(z,w) = \text{minlength}(c) \text{ for all paths } c \text{ from } z \text{ to } w.$$

Theorem 16. *Given two points z and w, the paths of minimal length from z to w are precisely those that go along a geodesic (without ever turning back, of course).*

6.2 Areas

Definition 6.2. We define the hyperbolic area of a region $\mathbb{U} \subset \mathbb{H}$ by

$$\text{area}(\mathbb{U}) = \int_U \frac{1}{y^2} dxdy.$$

Theorem 17. *The area of any hyperbolic triangle is $< \pi$.*

6.3 Triangle Geometry

Theorem 18 (Gauss-Bonnet Theorem)**.** *For a hyperbolic triangle T, with angles (α, β, γ),*

$$\text{area}(T) = \pi - \alpha - \beta - \gamma.$$

7 Hyperbolic Isomotries

7.1 Mobi$\ddot{u}$s Transformation

7.2 Hyperboic Isometries

Definition 7.1 (The Simplest Isometries)**.** A hyperbolic isometry is a map $\Phi : \mathbb{H} \to \mathbb{H}$ which is reversible, and compatible with all notions of hyperbolic geometry we have introduced. Namely, it
(1) preserves angles (in the sense of the angles between the tangent lines, at the intersection of two curves);
(2) preserves hyperbolic distances;
(3) preserves hyperbolic arclengths of curves;
(4) preserves hyperbolic areas of regions;
(5) takes hyperbolic geodesics to hyperbolic geodesics;
(6) takes hyperbolic circles to hyperbolic circles.

[Horizontal Translation]
[Vertical Translation]
[Hyperbolic Rotations]

Theorem 19. *Here's the general form of hyperbolic isometries. Take a real 2×2 matrix A, with det(A)>0. Each such matrix give rise to a hyperbolic isometry A, like this:*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ we have } \Phi_A(z) = \frac{az+b}{cz+d}.$$

8 Mostow's Theorem

Theorem 20 (Mostow Rigidity Theorem)**.** *Let M, N be closed hyperbolic manifolds of dimension at least 3, and let $f : M \to N$ be a homotopy equivalence. Then f is homotopic to an isometry.*

Proof. We prove this theorem following Gromov using the machinary of Gromov norms.

Since hyperbolic manifolds are $K(\pi, 1)$'s, two such manifolds M, N are homotopy equivalent if and only if their fundamental groups are isomorphic. Moreover, outer automorphisms of $\pi_1(M)$ induce self-homotopy equivalences of M. Since the group of isometries of a closed Riemannian manifold is a compact Lie group, it follows that $Out(\pi_1(M))$ is finite whenever M is closed and hyperbolic of dimension of at least 3. □

Definition 8.1 (Quasi-isometries)**.** Let $f : M \to N$ be a homotopic equivalence between closed hyperbolic manifolds, with homotopy inverse $g : N \to M$. We may assume these maps are smooth, and therefore Lipschitz. These lift to Lipschitz maps $\tilde{f} : \tilde{M} \to \tilde{N}$ and $\tilde{g} : \tilde{N} \to \tilde{M}$ between the universal covers (which are both isomorphic to $\mathbb{H}^n$) whose composition satisfies $d(\tilde{g}\tilde{f}(p), p) \leq C$ for some constant C independent of $p \in \tilde{M}$. It follows that $\tilde{f}$ (and likewise $\tilde{g}$) is a quasi-isometry; i.e. there exists a constant K so that for all $p, q \in \tilde{M}$ we have

$$\frac{1}{K} d_{\tilde{N}}(\tilde{f}(p), \tilde{f}(q)) - K \leq d_{\tilde{M}}(p, q) \leq K d_{\tilde{N}}(\tilde{f}(p), \tilde{f}(q)) + K. \tag{15}$$

If γ is a geodesic in $\mathbb{H}^n$, we can define a function $\rho : \mathbb{H}^n \to \mathbb{R}^+$ to be the distance to γ. Nearest point projection defines a retraction $\pi : \mathbb{H}^n \to \gamma$. If $S_t(\gamma)$ denotes the level set $\rho = t$, then $d\pi|TS_t$ is strictly contracting, with norm $\frac{1}{\sinh t}$. It follows that for every geodesic γ the image $\tilde{f}(\gamma)$ is contained within distance $O(\log(K))$ of some unique geodesic δ, and the map $\tilde{f}$ extends continuously (by taking the endpoints of γ to the endpoints of δ as above) to a homeomorphism $\tilde{f}_\infty : S^{n-1}_\infty \to S^{n-1}_\infty$, which intertwines the actions of $\pi_1(M)$ and $\pi_1(N)$ at infinity.

Definition 8.2 (Gromov norm)**.** For a (singular) homology class $\alpha \in H_k(X;\mathbb{R})$, the Gromov norm of α, denoted $||\alpha||$, is the infimum of $||z||$ over all real k-cycles z representing α.

Theorem 21 (Gromov proportionality)**.** *Let M be a closed, oriented hyperbolic n-manifold where $n \geq 2$. Then*

$$||[M]|| = \frac{volume(M)}{v_n} \tag{16}$$

where v_n is the supremum of the volumes of all geodesics n-simplices.

Proof. We first show that $||[M]|| \geq \frac{volume(M)}{v_n}$. This inequality will follow if we can show that for any cycle $\sum t_i\sigma_i$ there is homologous cycle $\sum t'_i\sigma'_i$ where every $\sigma'_i : \triangle_i \to M$ is totally geodesic, and $\sum |t_i| \geq \sum |t'_i|$. In fact, one can make this association functorial, by constructing a chain map $s : C_*(M,\mathbb{R}) \to C_*(M,\mathbb{R})$ taking simplices to geodesic simplices, which is chain homotopic to the identity.

The map s is defined on singular simplices $\sigma : \triangle_i \to M$ as follows. First, lift σ to a map to the universal cover $\tilde{\sigma} : \triangle_n \to \mathbb{H}^n$ where the link of $\mathbb{H}^n$ as the hyperboloid sitting in $\mathbb{R}^{n+1}$. The map $\tilde{\sigma}$ can be straightened to a linear map $\triangle_n \to \mathbb{R}^{n+1}$, and (radially) projected to a totally geodesic simplex in $\mathbb{H}^n$ (this is called the barycentric parametrization of the geodesic complex). Finally, this totally geodesic simplex can be projected back down to M, and the result is $s(\sigma)$. Eventually s is a chain map. Using the linear structure on $\mathbb{R}^n$ gives a canonical way to interpolate between id and s, and shows that s is chain homotopic to the identity, so induces the identity map on homology. This proves the first inequality.

We next show that $||[M]|| \leq \frac{volume(M)}{v_n}$, thereby completing the proof. It will suffice to exhibit a cycle $\sum t_i\sigma_i$ representing $[M]$ and with all t_i positive, for which each $\sigma_i(\triangle_n)$ is totally geodesic, with volume arbitrarily close to v_n.

Let $\triangle$ denote an isometry class of totally geodesic hyperbolic n-simplex with $|v_n - volume(\triangle')| < \epsilon$. The group $\text{Isom}(\mathbb{H}^n)$ acts transitively with compact point stabilizers on the space $D(\triangle)$ of isometric maps from $\triangle$ to $\mathbb{H}^n$, and we can put an invariant locally finite measure μ as a "measurable" singular n-chain in M, where by convention we parametrize each $\triangle$ by the standard simplex with a barycentric parametrization in such a way that the map to M is orientation-preserving. In fact, this space is really a (measurable) n-cycle, since for each $\triangle \to M$ and each face φ of $\triangle$ there is another isometric map $\triangle \to M$ obtained by reflection in φ, and the contributions of these two maps to φ under the boundary map will cancel. One can in fact develop the theory of Gromov norms for measurable homology, but it is easy enough to approximate this "measurable" chain by an honest geodesic singular chain whose simplices are nearly isometric to $\triangle$.

Choose a basepoint $p \in M$ and let p_1 denote a lift to the universal cover $\tilde{M} = \mathbb{H}^n$. Let E

be a compact fundamental domain for M, so that $\mathbb{H}^n$ is tiled by copies gE with $g \in \pi_1(M)$, each containing a single translate gp_1. For the sake of brevity, we denote $p_g := gp_1$. Now, if we denote an (n+1)-tuple $(g_0, ..., g_n) \in \pi_1(M)^{n+1}$ by $\vec{g}$ for short, we define $c(\vec{g})$ to be the μ-measure of the subset of $D(\triangle)$ consisting of isometric maps $\triangle \to \mathbb{H}^n$ denote the singular map sending the vertex i into g_iE. Furthermore, we let $\sigma_{\vec{g}} : \triangle_n \to \mathbb{H}^n$ denote the singular map sending the standard complex to the totally geodesic simplex with vertices p_{g_i}. The group $\pi_1(M)$ acts diagonally (from the left) on $\pi_1(M)^{n+1}$, and the projection $\pi \circ \sigma_{\vec{g}}$ is invariant under this action. We can therefore define a finite sum

$$z := \sum_{\vec{g} \in \pi_1(M) n \pi_1(M)^{n+1}} c(\vec{g}) \pi \circ \sigma_{\vec{g}} \tag{17}$$

which is a geodesic singular chain in $C_n(M;\mathbb{R})$ with all coefficients positive, and for which every simplex has volume at least $v_n - \epsilon$. Just as before z is actually a cycle, and represents a positive multiple of $[M]$. This proves the desired inequality, and the theorem. □

9 Kleinian Groups

10 Algebraic Convergence

11 The Techm$\ddot{u}$ller Space