HODGE THEORY ON RIEMANNIAN MANIFOLDS

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1. Introduction

The purpose of this paper is to present the basic ideas of Hodge theory on Riemannian manifolds and to develop the proof of the Hodge Decomposition Theorem. We begin in Section 2 by recalling the definitions of differential forms, the exterior derivative, and de Rham cohomology. In Section 3 we introduce the Hodge star operator and the codifferential, which allow us to define the Laplace–de Rham operator and the notion of harmonic forms. Section 4 is devoted to the proof of the Hodge Decomposition Theorem, which shows that every differential form can be written uniquely as the sum of an exact, coexact, and harmonic part. As a corollary, we deduce that each de Rham cohomology class has a unique harmonic representative.

2. Differential Forms

Before we introduce the main material some preliminaries must be developed, first of which *differential forms*. Simply put, differential forms are objects which we integrate on manifolds. Throughout this exposition, we follow the terminology and notation of [Lee06].

Definition 2.1. *Let $n \in \mathbb{N}$. A **smooth n-dimensional manifold** is a second-countable Hausdorff topological space M together with a maximal smooth atlas $\mathcal{A}$ such that each chart $(U, \varphi) \in \mathcal{A}$ satisfies:*

(1) $U \subset M$ is open.
(2) $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$.
(3) For any two charts $(U, \varphi), (V, \psi) \in \mathcal{A}$ with $U \cap V \neq \varnothing$, the transition map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is smooth (C^∞) between open subsets of $\mathbb{R}^n$.

On these manifolds we define something called the *tangent space*, which is the set of all tangent vectors to a point on a manifolds.

Definition 2.2. *Let M be a smooth n-dimensional manifold and $p \in M$. The **tangent space** at p, denoted T_pM, is the real vector space of* derivations *at p:*

$$T_pM := \{X : C^\infty(M) \to \mathbb{R} \mid X \text{ is linear and satisfies } X(fg) = X(f)g(p) + f(p)X(g)\}.$$

Each tangent vector $X \in T_pM$ represents an equivalence class of curves through p or, equivalently, a directional derivative at p.

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We call the disjoint union of all of the tangent spaces the *tangent bundle* which we denote

$$TM := \coprod_{p\in M} T_pM.$$

The *cotangent space* is the dual space to the tangent space and it is the vector space of linear maps

$$T_p^*M := \{l : T_pM \to \mathbb{R}\}.$$

An element $l \in T_p^*M$ is called a *covector*.

Definition 2.3. *Let T_p^*M be the cotangent space of the manifold M at the point p. The k^{th}* ***exterior power*** *of T_p^*M, denoted $\Lambda^k T_p^*M$, is the vector space consisting of all alternating multilinear maps*

$$\omega : \underbrace{T_pM \times \cdots \times T_pM}_{k\ times} \to \mathbb{R}.$$

Definition 2.4. *Given a smooth manifold M, a* ***differential k-form*** *is an assignment $p \mapsto \omega_p$ for a point $p \in M$ where $\omega_p \in \Lambda^k T_p^*M$ is an alternating linear map. The space of smooth differential k-forms on M is denoted $\Omega^k(M)$.*

In other words, a k-form takes k different tangent vectors at a point and outputs a real number. This means that 0-forms are just smooth functions, 1-forms are regular differentials (which are covectors) and 2-forms are area elements to integrate. For example, $f(x)\,dx$ is a *differential 1-form.*

We want to be able to generalize these 1-forms into k-forms and the way this is done is by using something called the *wedge product* or *exterior product*.

The wedge operator denoted $\wedge$ takes in 1-forms and spits out a higher dimensional form. Given k 1-forms $\omega_1, \omega_2, \dots, \omega_k \in \Omega^1(M)$ and k points in the tangent space $v_1, v_2, \dots, v_k$ the wedge product is defined as

$$(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k)(v_1, v_2, \dots, v_k) := \det \begin{bmatrix} \omega_1(v_1) & \cdots & \omega_k(v_1) \\ \vdots & dots & \vdots \\ \omega_1(v_k) & \cdots & \omega_k(v_k) \end{bmatrix}$$

To generalize we can write it in terms of a tensor product, defined as $(T_1 \otimes T_2)(v_1, v_2) := T_1(v_1) \cdot T_2(v_2)$. For k-form ω and l-form η we have the $(k+l)$-form

$$\omega \wedge \eta := \frac{1}{k!l!} \sum_{\sigma \in S_{k+1}} \operatorname{sgn}(\sigma)(\omega \otimes \eta) \circ \sigma.$$

where σ is a permutation of the $(k+l)^{\text{th}}$ symmetric group and sgn is the sign of the permutaton. This definition is the same as the alternating tensor for $\omega \otimes \eta$.

Lemma 2.1. *The following is true about wedge products if ω, η, ξ are differential forms.*

(a) $(a\omega + b\eta) \wedge \xi = a(\omega \wedge \xi) + b(\nu \wedge \xi)$.
(b) $\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi$.
(c) For k-form ω and l-form η, $\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega$.

It is in fact true that we can decompose all differential forms into the sum of wedge products.

Proposition 2.5. *For a k-form ω, given a local coordinate basis $\{x^i\}$, we can write*

$$\omega = \sum_{i_1<\cdots<i_k} f_I \, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}.$$

for 0-form f_I. Let $dx^I = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$.

Now that we have this, it is possible to define the derivative of a differential form, called the *exterior derivative*. This takes k-forms to $(k+1)$-forms.

Definition 2.6. *Let $\omega = \sum f_I \, dx^I$ then we denote the* ***exterior derivative*** *of ω as $d\omega$ and define it as*

$$d\omega := \sum df_I \wedge dx^I.$$

By plugging in and simplifying one can check that there exists a wedge product rule for the exterior derivative similar to the product rule from calculus.

Lemma 2.2. *For k-form ω and l-form η we have that $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.*

Lemma 2.3. *$d \circ d = 0$, i.e. $d(d\omega) = 0$ for every differential form ω.*

Proof. It suffices to check the identity locally in coordinates. First consider a 0-form (smooth function) f. In local coordinates $(x^1, \ldots, x^n)$

$$df = \sum_{i=1}^{n} \partial_i f \, dx^i,$$

so

$$d(df) = \sum_{i,j} \partial_j \partial_i f \, dx^j \wedge dx^i = \frac{1}{2} \sum_{i,j} \big(\partial_j \partial_i f - \partial_i \partial_j f \big) \, dx^j \wedge dx^i = 0,$$

since mixed partials commute.

For a general k-form $\omega = \sum_I f_I \, dx^I$ (summed over increasing multi-indices I), the exterior derivative acts only on the coefficient functions f_I, so

$$d(d\omega) = \sum_I d(df_I) \wedge dx^I = 0$$

by the 0-form case. Thus $d \circ d = 0$ on all forms. □

With the exterior derivative in hand, we can now distinguish between two important classes of differential forms.

Definition 2.7. *A k-form ω is called* ***closed*** *if $d\omega = 0$, and* ***exact*** *if there exists a $(k-1)$-form η such that $\omega = d\eta$.*

Since $d^2 = 0$, every exact form is automatically closed, but the converse need not hold. This leads us naturally to *de Rham cohomology*, which measures the "difference" between closed forms and exact forms. We denote closed and exact forms as $Z^k(M)$ and $B^k(M)$ respectively. Formally, we have

$$Z^k(M) = \ker\{d : \Omega^k(M) \to \Omega^{k+1}(M)\},$$

$$B^k(M) = \text{im}\{d : \Omega^{k-1}(M) \to \Omega^k(M)\}.$$

From before we know that $B^k(M) \subseteq Z^k(M)$ so we define the de Rham group as the quotient.

Definition 2.8. *We say that the k^{th}* ***de Rham cohomology group*** *is the quotient vector space*

$$H^k_{\text{dR}}(M) = \frac{Z^k(M)}{B^k(M)}.$$

3. The Hodge Star Operator

So far we've been dealing with a general manifold M but for Hodge theory we want to focus on a specific type of manifold called a *Riemannian manifold*, which is a smooth manifold with a *Riemannian metric* on it. Our treatment of the Hodge star operator follows [Ros97].

Definition 3.1. *Let M be a smooth manifold. A* ***Riemannian metric*** *is a smooth assignment $g : p \mapsto g_p$ for $p \in M$ where each g_p is an inner product on the tangent space*

$$g_p : T_pM \times T_pM \to \mathbb{R}.$$

Definition 3.2. *A* ***Riemannian manifold*** *is a just a pair (M, g) of a smooth manifold and a Riemannian metric.*

This means that for a curve $\gamma : [0, 1] \to M$ we define the length of γ to be

$$\int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt.$$

Notice that for Euclidean space $\mathbb{R}^n$ the metric is just $g(u, v) = u \cdot v$.

Let (M, g) be a Riemannian manifold of dimension n, so that for each point $p \in M$ the metric g_p defines an inner product on the tangent space T_pM.

We now define a pointwise inner product for differential forms and then a global inner product, called the *L^2 inner product.*

Definition 3.3. *Given $p \in M$, let $dx^1, dx^2, \ldots, dx^n$ be an orthonormal basis of the cotangent space T^*_pM with respect to g. Then for*

$$\alpha = \sum_I a_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad \beta = \sum_I b_I dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k T^*_pM,$$

define the inner product

$$\langle \alpha, \beta \rangle_p := \sum_I a_I b_I.$$

Using the metric g, we obtain a globally defined volume form $\text{vol}_g \in \Omega^n(M)$, which satisfies

$$\text{vol}_g = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

Definition 3.4. *For M, define an inner product on the space of smooth k-forms $\Omega^k(M)$ by*

$$\langle \alpha, \beta \rangle := \int_M \langle \alpha_p, \beta_p \rangle_p \, \text{vol}_g.$$

We can now define the *Hodge star operator*, which takes k-forms to $(n-k)$-forms.

Definition 3.5 (Hodge star operator). *For each $p \in M$, the* ***Hodge star operator***

$$* : \Lambda^k T_p^* M \to \Lambda^{n-k} T_p^* M$$

is the unique linear map satisfying

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle_p \operatorname{vol}_g,$$

for all $\alpha, \beta \in \Lambda^k T_p^* M$.

The operator $*$ extends smoothly to all of $\Omega^k(M)$ by applying it pointwise.

Proposition 3.6. *The Hodge star satisfies*

$$* * \alpha = (-1)^{k(n-k)} \alpha,$$

for all $\alpha \in \Omega^k(M)$.

Definition 3.7. *Define the* ***codifferential*** *operator $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ by*

$$\delta := (-1)^{nk+n+1} * d *,$$

where d is the exterior derivative.

Lemma 3.1. *The codifferential δ is the adjoint of d with respect to the L^2 inner product, i.e., for all $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$,*

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle.$$

Proof. We have

$$\langle d\alpha, \beta \rangle = \int_M \langle d\alpha, \beta \rangle \operatorname{vol}_g = \int_M d\alpha \wedge *\beta.$$

Applying the product rule we get

$$\int_M d\alpha \wedge *\beta = \int_M d(\alpha \wedge *\beta) - (-1)^{k-1} \int_M \alpha \wedge d(*\beta).$$

By Stokes' Theorem the first term goes to 0 leaving us with

$$(-1)^k \int_M \alpha \wedge d * \beta.$$

By using Proposition 3.6 we can rewrite this to get

$$(-1)^{nk+n+1} \int_M \alpha \wedge *(*d * \beta) = \langle \alpha, \delta\beta \rangle.$$

□

Definition 3.8. *A form ω is called* ***coexact*** *if it can be written as $\omega = \delta\eta$.*

4. Hodge Decomposition

Now we can get to the Hodge Theory, which eventually builds to prove the *Hodge Decomposition Theorem*. The proof of the theorem will follow [War83]. We start by defining the *Laplace–de Rham operator*, a second order differential operator acting on k-forms which is a generalization of the ordinary Laplacian.

Definition 4.1. *The* ***Laplace–de Rham operator*** *is*

$$\Delta := d\delta + \delta d : \Omega^k(M) \to \Omega^k(M).$$

A k-form ω is harmonic *if $\Delta\omega = 0$. Denote $\mathcal{H}^k(M) = \{\omega \in \Omega^k(M) : \Delta\omega = 0\}$.*

Lemma 4.1. *Δ is self adjoint with respect to the L^2 inner product.*

Proof. This follows from Lemma 3.1. □

Lemma 4.2. *If $\omega \in \Omega^k(M)$ and $\Delta\omega = 0$, then $d\omega = 0$ and $\delta\omega = 0$.*

Proof. Compute $\langle \Delta\omega, \omega\rangle = \langle d\delta\omega, \omega\rangle + \langle \delta d\omega, \omega\rangle = \langle \delta\omega, \delta\omega\rangle + \langle d\omega, d\omega\rangle$. If $\Delta\omega = 0$ the left-hand side vanishes, hence both norms vanish, so $d\omega = \delta\omega = 0$. □

Say that we have a solution ω to the PDE $\Delta\omega = \alpha$. Then ω determines the functional $l : \Omega^k(M) \to \mathbb{R}$ defined as

$$l(\beta) := \langle \omega, \beta\rangle.$$

Due to the self adjointness we know that ω is a solution if l satisfies

$$l(\Delta\varphi) = \langle \alpha, \varphi\rangle \text{ for all } \varphi \in \Omega^k(M).$$

We call such a functional a *weak solution* to the equation. In fact it is true that every weak solution l actually determines an ordinary solution ω. This is a result called *elliptic regularity*, which we will assume as it's a long proof.

Theorem 4.2. *Let $\alpha \in \Omega^k(M)$ and let l be a weak solution of $\Delta\omega = \alpha$ then there exists $\omega \in \Omega^k(M)$ such that*

$$l(\beta) = \langle \omega, \beta\rangle$$

for every $\beta \in \Omega^k$.

We will also assume the following result.

Theorem 4.3. *Let $\{a_n\}$ be a sequence of smooth k-forms such that $\|a_n\| \le c$ and $\|\Delta a_n\| \le c$ for some constant $c > 0$. Then a subsequence of $\{a_n\}$ is Cauchy in $\Omega^k(M)$.*

Theorem 4.4 (Hodge Decomposition Theorem)**.** *We have the following orthogonal direct sum decomposition of $\Omega^k(M)$:*

$$\begin{aligned}\Omega^k(M) &= \Delta(\Omega^k(M)) \oplus \mathcal{H}^k \\ &= d\Omega^{k-1} \oplus \delta\Omega^{k+1} \oplus \mathcal{H}^k\end{aligned}$$

and $\Delta\omega = \alpha$ has a solution iff α is orthogonal to $\mathcal{H}^k$.

Proof. We have the space of harmonic forms $\mathcal{H}^k$ so we can orthogonally decompose Ω^k as

$$\Omega^k = \mathcal{H}^k \oplus (\mathcal{H}^k)^\perp.$$

We will prove the theorem by showing that $(\mathcal{H}^k)^\perp = \Delta(\Omega^k)$. We do that by showing both are subsets of each other. One direction is fairly simple. Say that we have $\omega \in \Omega^k$ and $\alpha \in \mathcal{H}^k$. Then

$$\langle \Delta\omega, \alpha\rangle = \langle \omega, \Delta\alpha\rangle = 0.$$

meaning $\Delta(\Omega^k) \subset (\mathcal{H}^k)^\perp$. It remains to show the reverse inclusion

$$(\mathcal{H}^k)^\perp \subset \Delta(\Omega^k).$$

To solve $\Delta\omega = \alpha$ for a given $\alpha \in (\mathcal{H}^k)^\perp$, we want to define a linear functional on $\Delta(\Omega^k)$ by

$$L(\Delta\varphi) := \langle \alpha, \varphi\rangle.$$

For this to make sense we need L to be bounded and depend only on $\Delta\varphi$. This will ensure that $|L(\Delta\varphi)| \leq C\|\alpha\| \cdot \|\Delta\varphi\|$, which is exactly what allows us to extend L and represent it by an inner product with a smooth form.

We claim that there exists $c > 0$ such that

$$\|\beta\| \leq c\,\|\Delta\beta\| \qquad \text{for all } \beta \in (\mathcal{H}^k)^\perp.$$

Suppose this is false. Then there is a sequence $\beta_j \in (\mathcal{H}^k)^\perp$ with $\|\beta_j\| = 1$ and $\|\Delta\beta_j\| \to 0$. By Theorem 4.3, a subsequence is Cauchy in $\Omega^k(M)$, so for every $\psi \in \Omega^k(M)$ the limits

$$l(\psi) := \lim_{j\to\infty} \langle \beta_j, \psi\rangle$$

exist and define a bounded linear functional l on $\Omega^k(M)$ with $\|l\| \leq 1$. For any $\varphi \in \Omega^k(M)$ we compute

$$l(\Delta\varphi) = \lim_{j\to\infty} \langle \beta_j, \Delta\varphi\rangle = \lim_{j\to\infty} \langle \Delta\beta_j, \varphi\rangle = 0,$$

so l is a weak solution of $\Delta\omega = 0$. By Theorem 4.2, there is a smooth $\beta \in \Omega^k(M)$ with $\Delta\beta = 0$ such that $l(\psi) = \langle \beta, \psi\rangle$ for all ψ. Since each $\beta_j \in (\mathcal{H}^k)^\perp$, we have $\langle \beta, \beta\rangle = \lim\langle \beta_j, \beta\rangle = 0$, so $\beta = 0$. But then $\|\beta_j\| \to \|\beta\| = 0$, contradicting $\|\beta_j\| = 1$.

Now fix $\alpha \in (\mathcal{H}^k)^\perp$ and define $L : \Delta(\Omega^k) \to \mathbb{R}$ by

$$L(\Delta\varphi) := \langle \alpha, \varphi\rangle.$$

This is well defined: if $\Delta\varphi_1 = \Delta\varphi_2$, then $\varphi_1 - \varphi_2 \in \mathcal{H}^k$ so $\langle \alpha, \varphi_1 - \varphi_2\rangle = 0$. To see L is bounded, write $\varphi = \varphi^\perp + H(\varphi)$ with $\varphi^\perp \in (\mathcal{H}^k)^\perp$ and $H(\varphi) \in \mathcal{H}^k$ ($H(\varphi)$ is the harmonic projection of φ). Then

$$L(\Delta\varphi) = \langle \alpha, \varphi^\perp\rangle, \qquad \Delta\varphi = \Delta\varphi^\perp.$$

By Cauchy-Schwarz,

$$|L(\Delta\varphi)| \leq \|\alpha\|\,\|\varphi^\perp\| \leq c\,\|\alpha\|\,\|\Delta\varphi^\perp\| = c\,\|\alpha\|\,\|\Delta\varphi\|.$$

Thus L is bounded on $\Delta(\Omega^k)$. By Hahn–Banach, L extends to a bounded linear functional $\widetilde{L}$ on all of $\Omega^k(M)$. By Theorem 4.2, there exists $\omega \in \Omega^k(M)$ such that

$$\widetilde{L}(\psi) = \langle \omega, \psi \rangle \quad \text{for all } \psi \in \Omega^k(M).$$

In particular, for any $\varphi \in \Omega^k(M)$ we have

$$\langle \Delta\omega, \varphi \rangle = \langle \omega, \Delta\varphi \rangle = \widetilde{L}(\Delta\varphi) = L(\Delta\varphi) = \langle \alpha, \varphi \rangle.$$

Since this holds for all φ, we conclude $\Delta\omega = \alpha$. Therefore $(\mathcal{H}^k)^\perp \subset \Delta(\Omega^k)$.

Combining both inclusions gives the orthogonal decomposition

$$\Omega^k(M) = \Delta(\Omega^k(M)) \oplus \mathcal{H}^k.$$

It is now clear that $\Delta\omega = \alpha$ has a solution if and only if α is orthogonal to $\mathcal{H}^k$, i.e. the harmonic component of α vanishes. This completes the proof. □

This actually says something about the de Rham cohomology of M.

Corollary 4.4.1. *Each de Rham cohomology class has a unique harmonic representative. In particular,*

$$H^k_{\mathrm{dR}}(M) \cong \mathcal{H}^k(M).$$

Proof. Let $\alpha \in \Omega^k(M)$ be closed. By the Hodge decomposition theorem we may write

$$\alpha = d\beta + \delta\gamma + h$$

with $h \in \mathcal{H}^k(M)$. Applying d, we find

$$0 = d\alpha = d\delta\gamma,$$

since $dd = 0$ and $dh = 0$. The operator $d\delta$ maps $(k+1)$-forms to exact k-forms, so $d\delta\gamma$ is exact. Hence $\delta\gamma$ itself represents the trivial class in cohomology, and we conclude that α and h differ by an exact form. Thus every cohomology class has a harmonic representative.

Uniqueness follows from orthogonality: if $h, h' \in \mathcal{H}^k(M)$ are cohomologous, then $h - h' = d\beta$ for some β. But $h - h'$ is harmonic, hence orthogonal to all exact forms, so

$$\langle h - h', d\beta \rangle = \|h - h'\|^2 = 0.$$

Therefore $h = h'$. This proves the isomorphism $H^k_{\mathrm{dR}}(M) \cong \mathcal{H}^k(M)$. □

References

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