Hyperbolic Geometry

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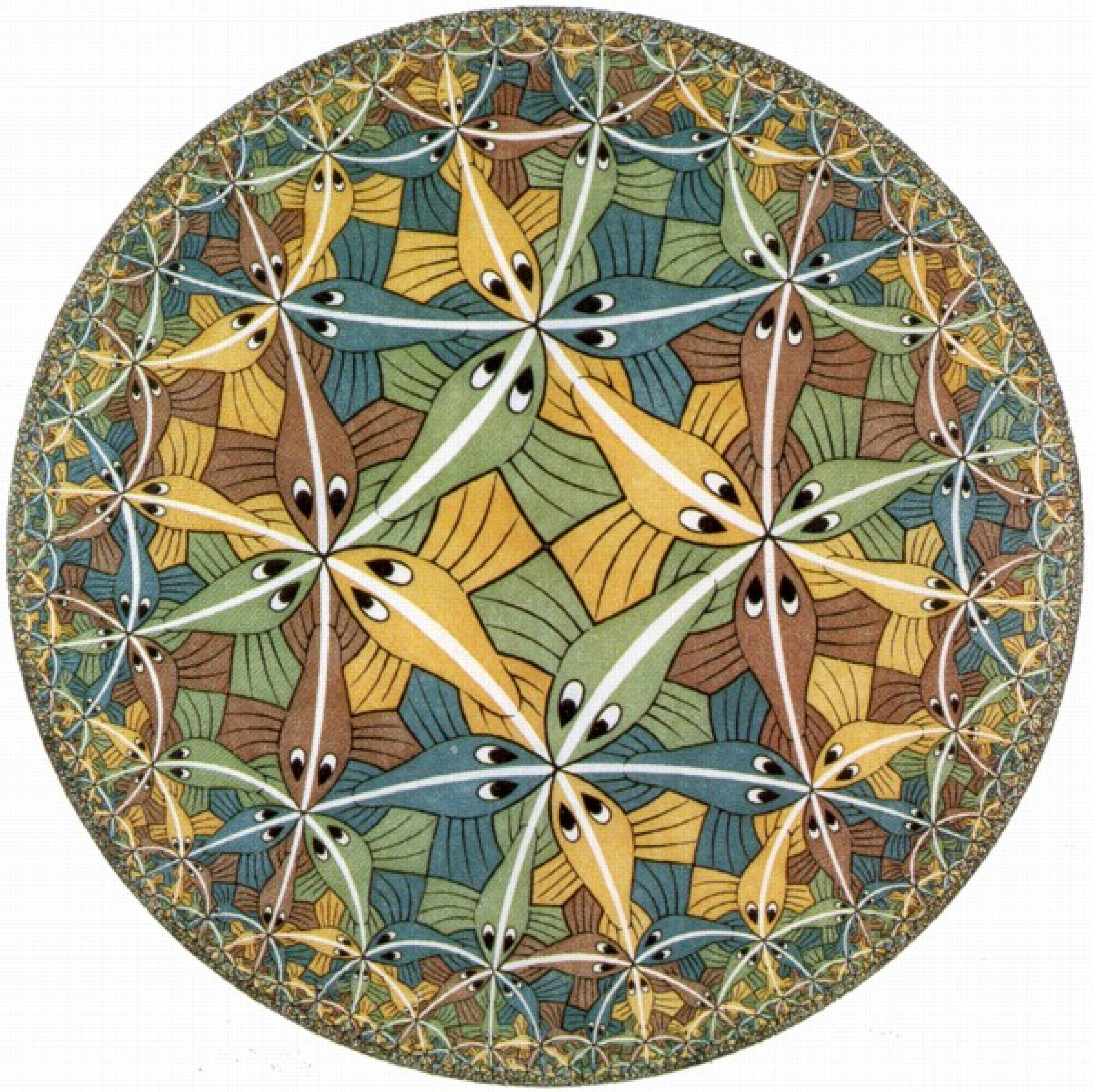

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HYPERBOLIC GEOMETRY

SHIVEN UPPAL

Abstract. Hyperbolic geometry is a non-Euclidean geometry where geodesics play the role of straight lines. It is characterized by a constant negative curvature. The idea was conceived when mathematicians tried to understand the parallel postulate. Hyperbolic geometry finds applications in mathematics, physics, and art, offering insights into the structure of space, universe, and tessellations. This paper introduces the foundations of hyperbolic geometry, geometric interpretation of curvature, the different foundational hyperbolic models, mathematical theorems, and the application of hyperbolic geometry.

1. Introduction

Euclidean geometry dictated mathematical thinking for two millennia with its five postulates. For centuries, mathematicians tried to prove the parallel postulate using the first four but failed. Gauss, Bolyai, and Lobachevsky negated the fifth postulate, and a new geometry stemmed from the failure of Euclid's fifth postulate known as **Hyperbolic Geometry**. Gauss made the remarkable discovery that denying *fifth postulate* could lead to a new geometry called the *Non-Euclidean Geometry*. He assumed that the sum of angles of a triangle is less than 180° denied the fifth postulate. Bolyai, and Lobachevsky independently published papers on Non-Euclidean Geometry providing strong evidence for its consistency, highlighting duality between non-Euclidean and spherical trigonometry where the hyperbolic trigonometric functions used in non-Euclidean play the same role that regular trigonometry does in non-Euclidean geometry.

2. Origin of Hyperbolic Geometry

2.1. **Euclid's Postulates:** 1

(1) A straight line segment can be drawn by joining any two points.
(2) Any straight line segment can be extended indefinitely in a straight line.
(3) Given any straight-line segment, a circle can be drawn, with the segment as a radius and one endpoint as the center.
(4) All right angles are congruent.

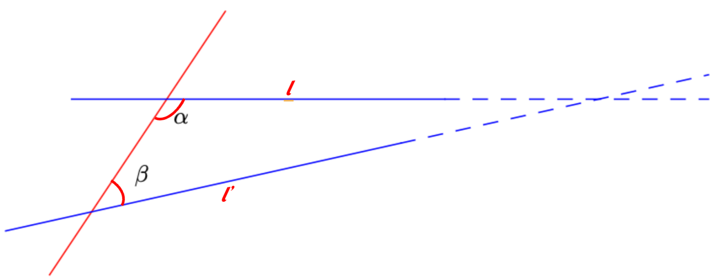

Figure 1. The fifth postulate: If $\alpha + \beta < 2R$, $(R = 90°)$ then ℓ and ℓ' intersect on the right side

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(5) If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the parallel postulate.

For centuries, scientists tried to prove the fifth postulate or the **parallel postulate** from the first four, but were forced to make additional assumptions to conclude it. During this period, Gauss, Lobachevsky, and Bolyai, independently developed a new type of geometry, now known as the Hyperbolic Geometry, which directly contradicted the parallel postulate of Euclidean geometry. **Lobachevsky** introduced the concept of the **angle of parallelism**: for a line l and a point A, a perpendicular distance a from it, the angle of parallelism α is the smallest angle such that the line l drawn from A, remains parallel to l, that is, it does not intersect l.

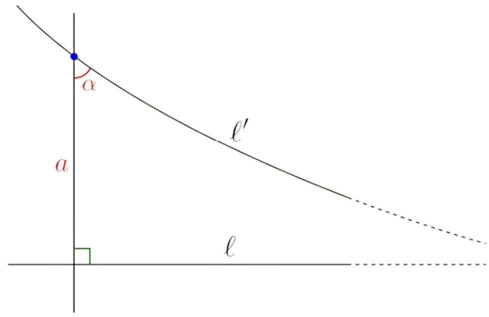

FIGURE 2. Angle of Parallelism

2.2. **Spherical-Hyperbolic Contrasts:** Spherical and hyperbolic geometry are opposite in many ways, starting from the way they handle the parallel postulate. In spherical geometry, the opposite of a straight line is **great circle**.
In spherical geometry, the parallel postulate takes the form of
"Through a point not on a given line, there is no line parallel to the given line".
In hyperbolic geometry, the parallel postulate states that
"Through a point not on a given line, there are infinitely many lines parallel to the given line".
The duality between spherical and hyperbolic geometry can be further explained by highlighting the contrasting properties of a triangle in each space.

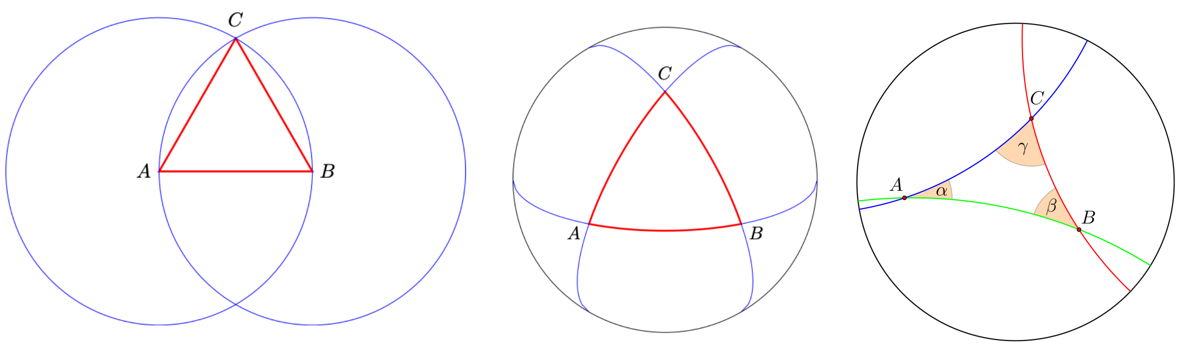

FIGURE 3. Euclidean Triangle, Spherical Triangle and Hyperbolic Triangle

- In **Euclidean geometry**, the angle sum is exactly $\alpha + \beta + \gamma = 180°$.
- In **spherical geometry**, the angle sum of a triangle satisfies $\alpha + \beta + \gamma > 180°$.
- In **hyperbolic geometry**, the angle sum obeys $\alpha + \beta + \gamma < 180°$.

Table 1. Comparison of Spherical, Euclidean, and Hyperbolic Geometry

Property	**Spherical Geometry**	**Euclidean Geometry**	**Hyperbolic Geometry**
Lines	Great circles that eventually converge	Infinite straight lines that never meet	Curved lines that diverge from one another
Sum of angles in a triangle	$\alpha+\beta+\gamma>180^\circ$	$\alpha+\beta+\gamma=180^\circ$	$\alpha+\beta+\gamma<180^\circ$
Number of parallels through a point not on a line	0	1	Infinite
Circumference of a circle	$2\pi\sin r$	$2\pi r$	$2\pi\sinh r$
Area of a circle	$2\pi(1-\cos r)$	πr^2	$2\pi(\cosh r-1)$
Pythagorean formula	$\cos c=\cos a\cos b$ (for right triangle)	$a^2+b^2=c^2$	$\cosh c=\cosh a\cosh b$ (for right triangle)
Area of triangle	$k((\alpha+\beta+\gamma)-\pi)$	Not proportional to angles	$k(\pi-(\alpha+\beta+\gamma))$
Parallel postulate	Does not hold (no parallels)	Holds	Does not hold (multiple parallels)
Force analogy	Squishing effect (like on a sphere)	No distortion	Tidal effect (stretching in opposite directions)
Tiling of space	Only certain polygons tile the surface	Regular tilings like squares, triangles, hexagons	Many tilings possible, even with heptagons, octagons etc.
Geodesics	Great circles	Straight lines	Arcs, semicircles, or rays (depending on model)
Model examples	Surface of Earth, $\mathbb{S}^2$	$\mathbb{R}^2$, the flat plane	Poincaré disk, half-plane, hyperboloid

3. Curvature

3.1. **What is Curvature?** The fundamental geometric property of a space that establishes the relationship between the angles of a triangle and its area is called **curvature**. It also establishes the relationship between the circumference of a circle and its radius. Curvature measures the extent to which the space is curved and in which direction it is curved. Consider a smooth curve. The curvature at any point of the curve quantifies how sharply it bends away from the tangent line. The *radius of curvature* at that point refers to the radius of the best-fitting circle called the *osculating circle* that approximates the circle locally. The curvature of the circle is the reciprocal of the radius and is given by:

$$\kappa = \frac{1}{r}$$

For surfaces, curvature measures how much the surface deviates from the tangent plane at a given point. In two dimensions, surfaces of constant curvature κ are categorized as follows:

- **Positive curvature** ($\kappa > 0$)**:** The surface lies entirely on one side of the tangent plane at any point (e.g. sphere). If the surface is simply connected, it is a sphere with curvature.
- **Negative curvature** ($\kappa < 0$)**:** The surface is saddle-shaped and the tangent plane intersects the surface on both sides (e.g. a hyperbolic paraboloid). A simply connected surface with negative curvature is the hyperbolic plane (or 2-dimensional hyperbolic space).
- **Zero curvature** ($\kappa = 0$)**:** The surface is flat in at least one direction and the straight line lies in the tangent plane at every point (e.g. a cylinder). A simply connected surface with zero curvature is the Euclidean plane.

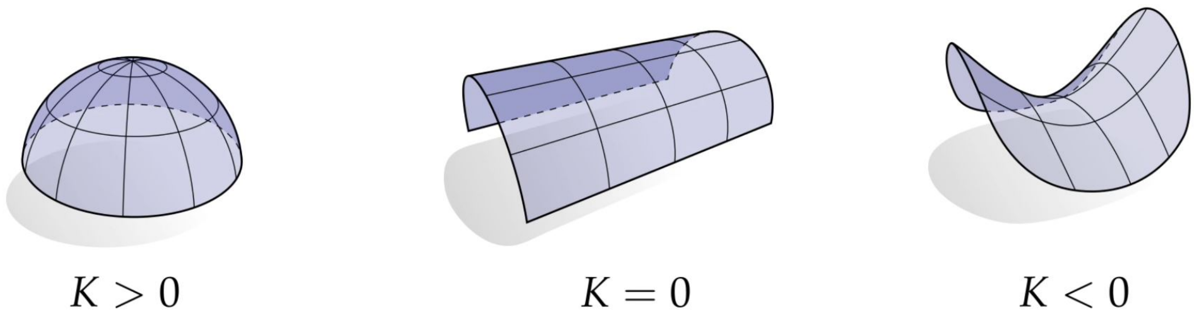

Figure 4. Interpreting the Gaussian curvature's value

Theorem 3.1 (Poincaré-Koebe Uniformisation Theorem)**.** *Let S be a compact, orientable surface with a constant curvature and no boundary. Then there exists a covering space M, with an appropriate distance function and a discrete group of isometries γ acting on M, such that the surface S is homeomorphic to the space γ. Depending on the curvature of S, the model space M is:*

- *A sphere, if S has a positive curvature,*
- *The Euclidean plane $\mathbb{R}^2$, if S has zero curvature,*
- *The hyperbolic plane $\mathbb{H}$, if S has a negative curvature.*

3.2. **Geometric Interpretation of Curvature:** Take a unit speed curve $\gamma(t)$ in $\mathbb{R}^2$. When we move from t to $t + \Delta t$, the curve deviates from its tangent line at $\gamma(t)$. The perpendicular distance from a point on the curve o this tangent line is given by

$$(\gamma(t+\Delta t)-\gamma(t))\cdot\mathbf{n},$$

where $\mathbf{n}$ is a unit vector perpendicular to the tangent vector $\dot{\gamma}(t)$ at the point $\gamma(t)$. By Taylor expansion, we have:

$$\gamma(t+\Delta t)=\gamma(t)+\dot{\gamma}(t)\Delta t+\frac{1}{2}\ddot{\gamma}(t)(\Delta t)^2+O((\Delta t)^3).$$

Since $\dot{\gamma}(t)\cdot\mathbf{n}=0$ (as n s normal to the tangent), the first-order term vanishes when projected onto $\mathbf{n}$, and we obtain the following approximation for small Δt:

$$(\gamma(t+\Delta t)-\gamma(t))\cdot\mathbf{n}\approx\frac{1}{2}\ddot{\gamma}(t)\cdot\mathbf{n}(\Delta t)^2.$$

This shows that the component of the second derivative in the direction normal to the curve gives a measure of how much the curve deviates from its tangent line.

Definition 3.1. *Let $\gamma(t):I\to\mathbb{R}^3$ be a curve parametrized by arc length $s\in I$. The number $|\ddot{\gamma}(t)|=\kappa(s)$ is called the curvature of γ at s.*

3.3. **Key properties of surfaces with Negative Curvature:**

- **Local saddle-shape behavior:** A surface with negative Gaussian curvature is locally non-convex and lies on both sides of tangent plane, exhibiting saddle-like geometry. In other words, a surface with **negative Gaussian Curvature** bends in opposite directions at a point (like a saddle) making the surface **non-convex** and lies partly above and below its tangent plane.
- **Thin Triangles** Geodesic triangles on negatively curved surfaces satisfy a distance comparison of the form

$$d(z,m)^2\le\tfrac{1}{2}\Big[d(z,x)^2+d(z,y)^2\Big]-\tfrac{1}{4}d(x,y)^2,$$

  where m is the midpoint of the side xy, indicating geometry of a "thin" triangle.

4. Classical Models of Hyperbolic Geometry

4.1. **Hyperboloid Model.** The hyperbolic model, also called the Minkowski model, is an n-dimensional model in which each point in the hyperbolic space is shown as a point on the upper sheet of the curved surface called a two-sheeted hyperboloid.

Definition 4.1. *Let*

$$\mathbb{H}=\Big\{(x,y,z)\in\mathbb{R}^3\;\Big|\;x^2+y^2-z^2=-1\Big\}=\Big\{p\in\mathbb{R}^{2,1}\;\Big|\;\langle p,p\rangle=-1\Big\}\subset\mathbb{R}^{2,1}$$

be the two-sheeted hyperboloid in Minkowski space $\mathbb{R}^{2,1}$. Then the upper sheet of the hyperboloid is defined as

$$H^+=H\cap\{z>0\}.$$

Metric: The induced metric on H^+ comes from restricting the Minkowski inner product to the tangent plane $T_{\boldsymbol{p}}H^+=\{\boldsymbol{p}\}^\perp$, resulting in a metric (positive-definite) on H^+. In coordinates, this Lorentzian metric is given by:

$$ds^2=dx^2+dy^2-dz^2.$$

Distance Function: The distance between two points $\boldsymbol{p},\boldsymbol{q}\in H^+$ is defined using the Lorentzian inner product: $d(\boldsymbol{p},\boldsymbol{q})=\operatorname{arcosh}(-\langle\boldsymbol{p},\boldsymbol{q}\rangle)$

This reflects the hyperbolic angle between the vectors

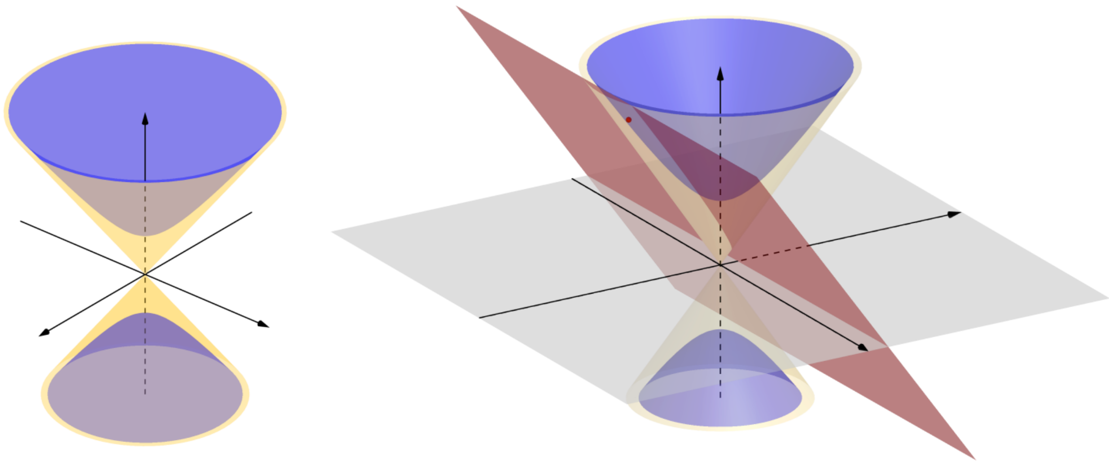

FIGURE 5. (a)The hyperboloid $\mathbb{H}$ in $\mathbb{R}^{n,1}$ (b) Tangent space to the hyperboloid at a point.

Geodesics: Geodesics in the hyperboloid model are curves that locally minimize distance. They correspond to the intersection of the hyperboloid with planes through the origin:

$$\gamma = H^+ \cap P,$$

where P is a 2-dimensional linear subspace of $\mathbb{R}^{2,1}$. Each geodesic can be parametrized as follows:

$$\gamma(t) = \cosh(\|\boldsymbol{v}\|t)\, \boldsymbol{p} + \sinh(\|\boldsymbol{v}\|t)\, \boldsymbol{v},$$

for some $\boldsymbol{p} \in H^+$ and $\boldsymbol{v} \in T_{\boldsymbol{p}}H^+$, the tangent space at $\boldsymbol{p}$.

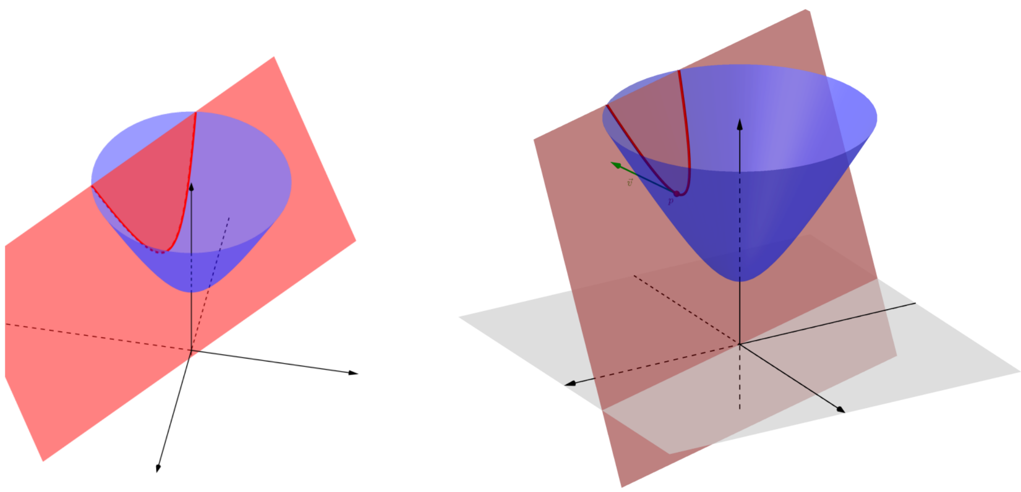

FIGURE 6. Geodesics on the Hyperboloid Model

Isometries: The isometry group of the hyperboloid model is the subgroup of Lorentz transformations that preserve the hyperboloid and its upper sheet:

$$O^+(2,1),$$

i.e., the group of linear transformations preserving the Minkowski inner product and the condition $z > 0$. The orientation-preserving subgroup is:

$$SO^+(2,1) = O_0(2,1).$$

4.2. **The Cayley–Klein Model (Projective Model).** The Cayley–Klein model, also known as the *Projective Model*, gives a projective representation of hyperbolic space. It defines the notion of distance as

$$\mathbb{H}^2 = (x, y) : x^2 + y^2 < 1$$

Metric. The metric on the Klein disk is:

$$ds^2 = \frac{dx^2 + dy^2}{1 - x^2 - y^2} + \frac{(x\,dx + y\,dy)^2}{(1 - x^2 - y^2)^2}.$$

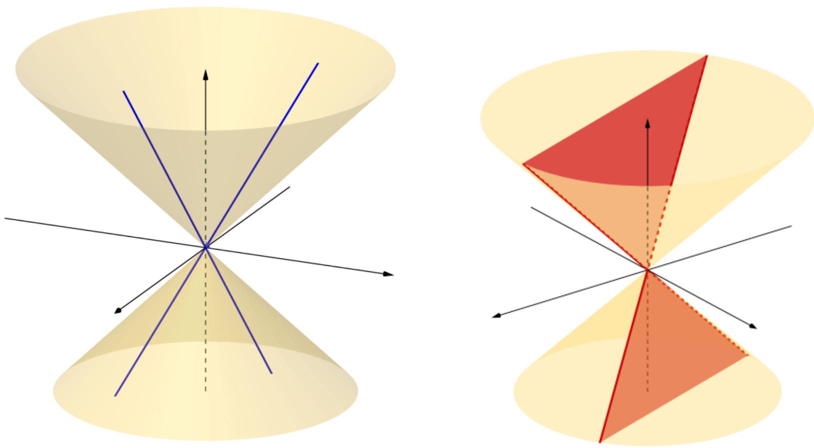

FIGURE 7. Cayley–Klein Model

Geodesics. Geodesics are staright chords of unit disk that minimise the hyperbolic distance.

Distance Formula. The distance from A to B is

$$d(A, B) = \frac{1}{2} \log \left(\frac{|AX|}{|BX|} : \frac{|AY|}{|BY|} \right)$$

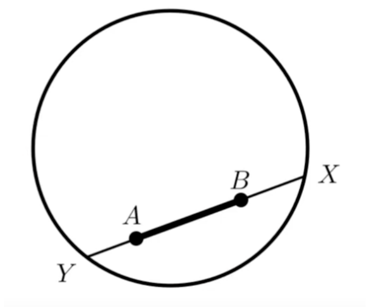

FIGURE 8. Distance between A and B in Cayley-Klein Model

Isometries. Isometries in the Cayley-Klein Model are projective transformations that preserve the unit disk and the metric. They are represented as Möbius transformations acting on homegeneous coordinates. Key isometries include translations, rotations and reflections.

4.3. **The Beltrami–Klein Model (Klein Disk).** The Beltrami–Klein model represents hyperbolic space as a subset of the Euclidean plane.

Definition 4.2.

$$\{x^2 + y^2 - z^2 < 0\} \cap \{z = 1\} \subset \mathbb{R}^3 \quad \approx \quad D = \{x^2 + y^2 < 1\} \subset \mathbb{R}^2,$$

i.e., the open unit disk in $\mathbb{R}^2$*.*

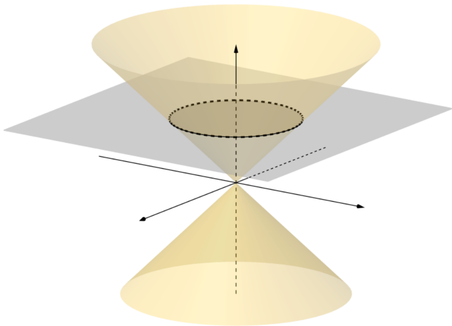

FIGURE 9. Beltrami-Klein Model

Metric. The metric on the Klein disk is:

$$ds^2 = \frac{dx^2 + dy^2}{1 - x^2 - y^2} + \frac{(x\,dx + y\,dy)^2}{(1 - x^2 - y^2)^2}.$$

Distance Formula. The hyperbolic distance between two points $\boldsymbol{p}, \boldsymbol{q} \in D$ is given by:

$$d(\boldsymbol{p}, \boldsymbol{q}) = \operatorname{arcosh}\left(\frac{1 - \langle \boldsymbol{p}, \boldsymbol{q}\rangle}{\sqrt{(1 - \|\boldsymbol{p}\|^2)(1 - \|\boldsymbol{q}\|^2)}}\right),$$

where:

- $\langle \boldsymbol{p}, \boldsymbol{q}\rangle$ is the standard Euclidean dot product,
- $\|\boldsymbol{p}\|$ is the Euclidean norm of $\boldsymbol{p}$.

Geodesics. Geodesics in the Beltrami–Klein model are **Euclidean straight-line chords** inside the unit disk D.

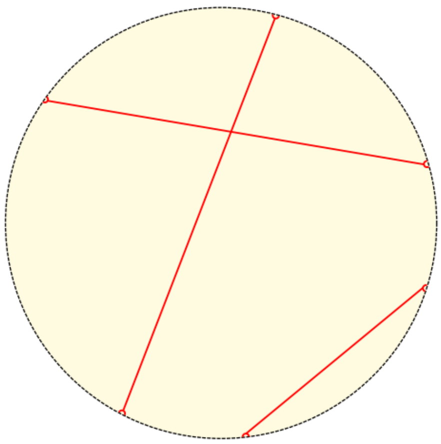

FIGURE 10. Beltrami-Klein Model Geodesics

Isometries. The isometry group is:

- $\mathrm{PO}(2,1)$, acting by projective (fractional linear) transformations on $\mathbb{R}^2$,
- Orientation-preserving subgroup:$\mathrm{PSO}(2,1)$.

4.4. Poincaré Disk Model.

Definition 4.3. *The Poincaré disk model or the conformal disk model represents the hyperbolic plane using the interior of the Euclidean unit circle:*

$$\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \quad | x^2 + y^2 < 1\}$$

and equivalently, as a subset of the complex plane:

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\} \subset \mathbb{C}$$

Metric: The metric on the disk is given by:

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} = \frac{4|dz|^2}{(1 - |z|^2)^2}$$

A hyperbolic line in the Poincaré disk model is either a Euclidean diameter of a unit circle or a circular arc that intersects the boundary circle at right angles. In the figure shown below, we have a hyperbolic line l and a point P such that many parallel lines to l pass through P.

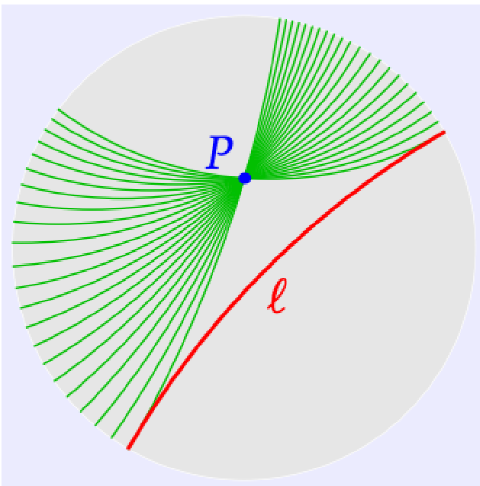

FIGURE 11. Parallel lines to hyperbolic line l and passing through point P

Lemma 4.1. *Every hyperbolic line in the Poincaré disk model is represented by one of the following:*

- ***Euclidean diameter*** *of the unit disk that passes through point* $(c, d) \neq (0, 0)$*, satisfying the Euclidean equation:* $dx = cy$
- *An* ***arc of a Euclidean circle*** *lying entirely within the unit disk and orthogonal to its boundary, given by the equation:*

$$x^2 + y^2 - 2ax - 2by + 1 = 0 \text{ where } a^2 + b^2 > 1$$

The circle has ***Center:*** $C = (a, b)$ *and* ***Radius:*** $r = \sqrt{a^2 + b^2 - 1}$

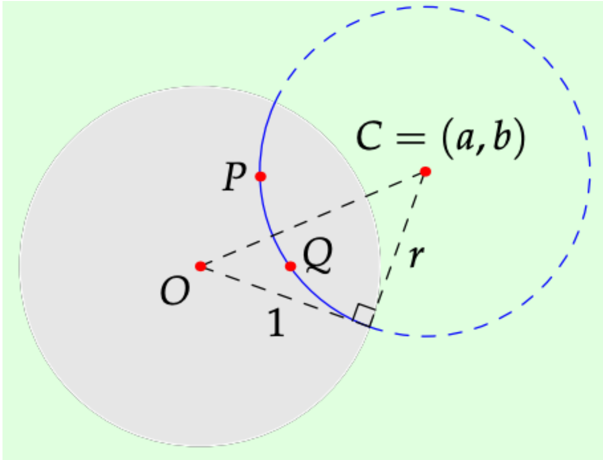

FIGURE 12. Hyperbolic line

Distance Formula: The *hyperbolic distance* between two points P and Q in the Poincaré disk is defined as:

$$d(P,Q) := \cosh^{-1}\left(1 + \frac{2|PQ|^2}{(1-|P|^2)(1-|Q|^2)}\right)$$

where $|PQ|$ denotes the Euclidean distance between P and Q, and $|P|$, $|Q|$ denote the Euclidean distance of P and Q from the origin, respectively.

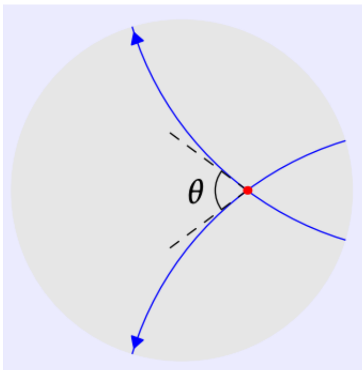

FIGURE 13. Hyperbolic distance

Two **hyperbolic segments** are said to be *congruent* if they have the same hyperbolic length. The **angle** between two hyperbolic rays is defined as the Euclidean angle between their tangent lines at the point of intersection. Angles are congruent if they have the same measure.

Lemma 4.2. *The **Hyperbolic distance** from a point P to the origin in Poincaré disk model is given by:*

$$d(O,P) = \cosh^{-1}\left(\frac{1+|P|^2}{1-|P|^2}\right) = \ln\left(\frac{1+|P|}{1-|P|}\right)$$

where $|P|$ denotes the Euclidean distance of the P from the origin.

Geodesics. The **geodesics** in the Poincaré disk are all the Euclidean circle arcs (including diameters) that intersect the boundary circle $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ orthogonally.

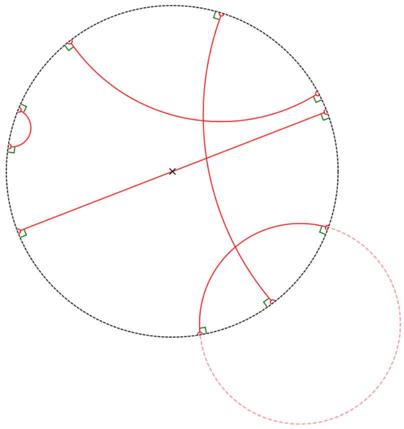

Figure 14. Geodesics on the Poincaré Disk Model

Isometries. The group of orientation-preserving **isometries** of the Poincaré disk is:

$$\mathrm{PSU}(1,1) = \mathrm{PU}(1,1)$$

acting by Möbius (fractional linear) transformations:

$$z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}, \quad \text{with } |a|^2 - |b|^2 = 1$$

Alternatively, an isometry can be written as:

$$z \mapsto u \cdot \frac{z-a}{1-\bar{a}z}, \quad \text{with } |u| = 1,\ |a| < 1$$

Orientation-reversing isometries replace z with $\bar{z}$.

4.5. **Poincaré Half-Plane Model.** The Poincaré half-plane model is one of the most commonly used models of hyperbolic geometry. There are infinitely many parallel lines to a given line l through a point P not on l.

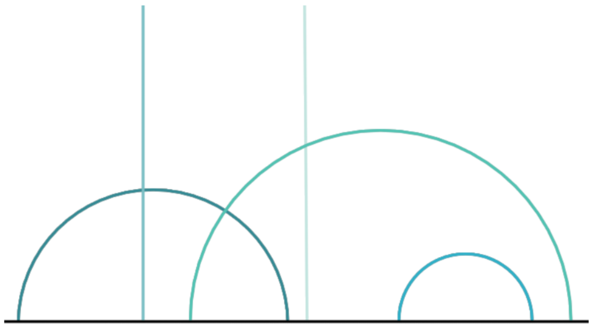

Figure 15. Upper Half-Plane Model

Definition 4.4. *Poincaré's upper half-plane denoted by $\mathbb{H}^2$ represents the upper half of the complex plane. It is described as:*

$$\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

The topology of $\mathbb{H}^2$ is induced from $\mathbb{R}^2$ where $\mathbb{R}^2$ is endowed with usual topology. Thus, $\mathbb{H}^2$ inherits the subspace topology from $\mathbb{R}^2$ or, it is a subspace in $\mathbb{R}^2$.

Metric. *To define, Poincaré metric, a metric on this upper half-plane,* $\mathbb{H}^2$*, the line element,* ds^2 *is denoted as:*

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{|dz|^2}{(\mathrm{Im}(z))^2}$$

It becomes singular as one approaches the real axis.

Suppose, we have a piecewise differentiable path $\alpha(t)$ in $\mathbb{H}^2$ expressed as

$$\alpha(t) = (x(t), y(t))$$

where $x(t)$ and $y(t)$ are functions of t, and $\alpha(t)$ lies in $\mathbb{H}^2$ so

$$\alpha(t) = (x(t), y(t)) \in \mathbb{H}^2 \subset \mathbb{R}^2$$

Since α is piecewise differentiable, both $x(t)$ and $y(t)$ are also piecewise differentiable. So, length of α, $L(\alpha)$ is

$$L(\alpha) = \int_a^b \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt$$

Since $x(t) and y$(t) are piecewise differentiable , $\frac{dx}{dt}$ and $\frac{dy}{dt}$ exist, making the integral defined, and giving formal definition of the length of path $\alpha(t)$ in hyperbolic geometry. Consider path $\alpha(t) = (0, t)$ where t ranges from a to b so $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 1$.

$$L(\alpha) = \int_a^b \frac{dt}{t} = \ln\left(\frac{b}{a}\right)$$

Thus, for a particular length α from $(0, a)$ to $(0, b)$ is given by $\ln(b/a)$. This length is referred to as the hyperbolic length of α. Complex notation:

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

$$ds^2 = \frac{|dz|^2}{(\mathrm{Im}(z))^2}$$

This is another way to express Poincaré metric. Here, dz represents the differential of the complex variable z, and it can be written as $dz = dx + idy$, where $z = x + iy$.

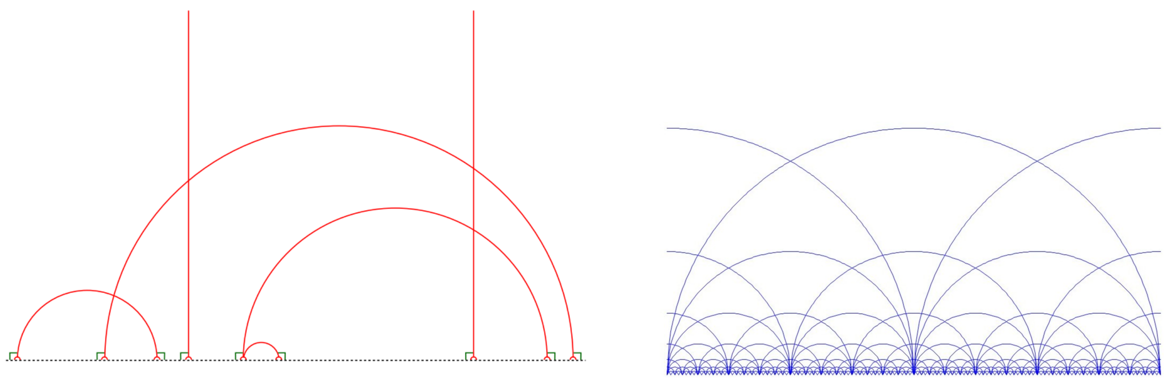

FIGURE 16. Geodesics on the Poincaré Half-Plane Model

Distance Formula. The distance between two points in the model is given by:

$$d(z_1, z_2) = \operatorname{arcosh}\left(1 + \frac{|z_1 - z_2|^2}{2y_1y_2}\right) = \ln[z_1, z_2, J, I]$$

This formula depends on the vertical positions y_1 and y_2 of the points.

Geodesics. Geodesics (shortest paths) are: Circular arcs orthogonal to the real axis, including vertical straight lines.

Isometries. The group of distance-preserving transformations is:

$$z \mapsto \frac{az+b}{cz+d}, \quad \text{with } a, b, c, d \in \mathbb{R},\ ad - bc = 1$$

These are fractional linear transformations from $\mathrm{PSL}(2, \mathbb{R})$. Orientation-reversing maps are obtained by conjugation.

4.6. **Comparison of the 5 models.** The following figure shows the relation between models in a hyperbolic space and the Geodesics in Poincaré, Klein and Hemisphere models.

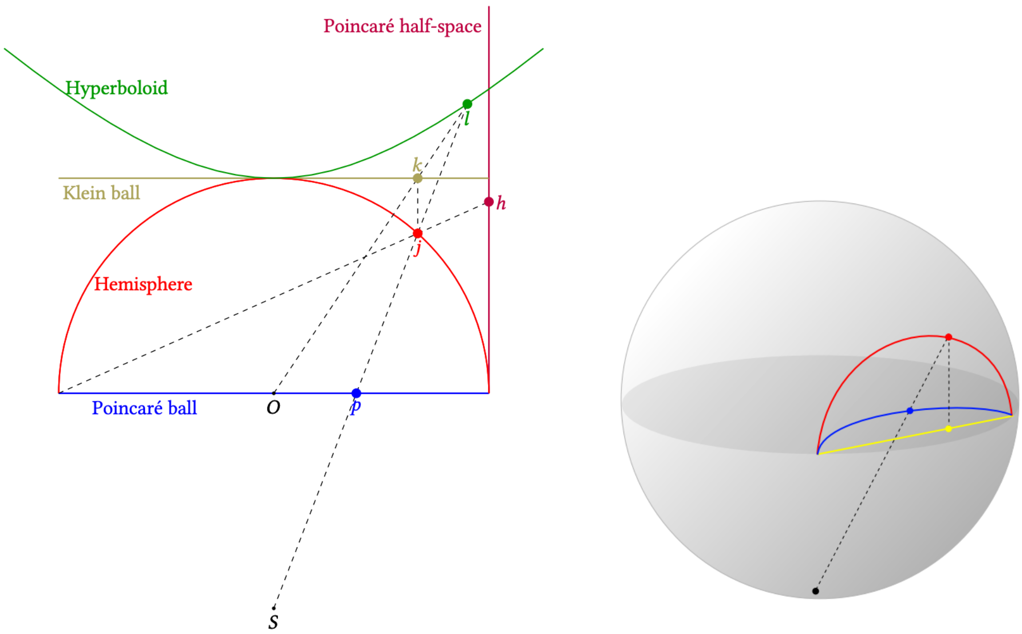

FIGURE 17. (a)Relation between models of hyperbolic space and Geodesics in Poincaré ball, Klein ball, and hemisphere models. (b)

5. Isometries and Distances in the Hyperboloid Model

5.1. **Hyperbolic Distance:** Let z and w, be two points belonging to the upper half-plane $\mathbb{H}^2$. The **hyperbolic distance** between these two points, denoted by $d_{\mathbb{H}^2}(z, w)$, is defined as the infimum (the greatest lower bound) of the lengths of all piecewise differentiable paths γ connecting z and w. This is expressed as:

$$d_{\mathbb{H}^2}(z, w) = \inf\{L(\gamma) \mid \gamma \text{ is a piecewise differentiable path from } z \text{ to } w$$

To tailor the distance in metric spaces, $d_{\mathbb{H}^2}$ to the hyperbolic context, we need to verify the three defining properties of a metric:

(1) **Non-negativity and identity of indiscernibles**: The distance between two points z and w in the upper half-plane is non-negative, and $d_{\mathbb{H}^2}(z, w) = 0$ if and only if $z = w$. This ensures that distinct points always have a positive distance between them.

(2) **Symmetry**: The distance function satisfies symmetry, meaning the distance from z to w is the same as the distance from w to z, or formally:

$$d^2_{\mathbb{H}}(z, w) = d_{\mathbb{H}^2}(w, z).$$

(3) **Triangle inequality**: The hyperbolic distance satisfies the triangle inequality, meaning that for any three points z, w, u in $\mathbb{H}^2$, we have:

$$d_{\mathbb{H}^2}(z, w) \leq d_{\mathbb{H}^2}(z, u) + d_{\mathbb{H}^2}(u, w).$$

This ensures that the direct distance between two points is always less than or equal to the sum of distances through any intermediate point. These properties together, prove that $d_{\mathbb{H}^2}$ is a valid metric, turning the upper half-plane $\mathbb{H}^2$ into a metric space.

The hyperbolic geometry described here violates the fifth postulate of Euclidean geometry. Consider the upper half-plane model of hyperbolic geometry. Assume two points, z and w, that lie on a vertical line in this upper half-plane. Join z and w by a vertical line segment and call this path α. Compare it to any other path γ, where γ is also assumed to be piecewise differentiable, then the length of α is always less than or equal to the length of γ. In other words, α, the vertical line segment, is the shortest path between z and w. This makes α the geodesic between these two points.

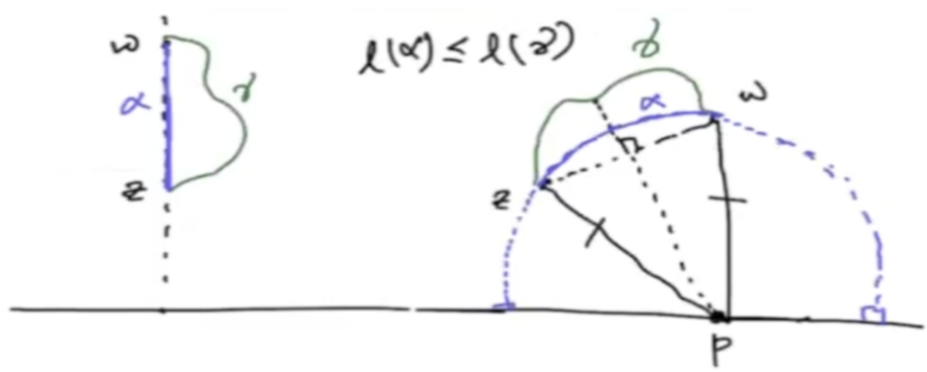

FIGURE 18. Shortest paths in hyperbolic geometry: vertical lines or portions of semicircles with center on Real axis

Next, consider the case where z and w do not lie on a vertical line. Join z and w by a Euclidean line segment and then draw the perpendicular bisector of this line that will meet the real axis (the x-axis) at some point, which we call p. The distances from p and z and from p and w will be equal in terms of Euclidean length. Using p as the center, we can now draw a semicircle that passes through both z and w, with a radius equal to the Euclidean distance from p and z.

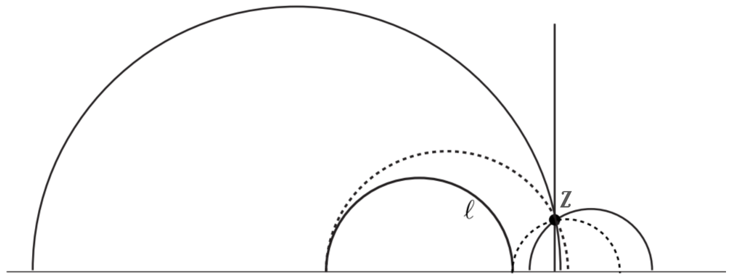

FIGURE 19. Violation of Fifth Axiom of Euclidean Geometry

In hyperbolic geometry, the geodesics are either vertical lines or arcs of semicircles centered on the real axis. These geodesics are the paths of the shortest distance between two points in the upper half-plane. Now, let us explore a fascinating property of hyperbolic geometry: if we take any point z in the upper half-plane and consider any geodesic that does not pass through z, it turns out there are infinitely many other geodesics that pass through z but do not intersect the original geodesic.

Geodesics are the paths of shortest distance between two points in the upper half-plane. They are either vertical lines or arcs of semicircles centered on the real axis. Take any point z in the upper half-plane and consider any geodesic that does not pass through z , it turns out there are infinitely many other geodesics that pass through z but do not intersect the original geodesic, l. If l is a vertical line, infinitely many semicircles centered on the real axis pass through z without intersecting l In hyperbolic geometry, however, there are infinitely many such lines, or in this case, geodesics, that do not intersect the given geodesic l.

5.2. **Isometry:**

Definition 5.1. *Let $T : \mathbb{H}^2 \to \mathbb{H}^2$ be a map expressed as $T(x, y) = u(x, y), v(x, y)$. For T to be an* ***isometry****, it must preserve hyperbolic metric and the following must hold:*

$$\frac{dx^2 + dy^2}{y^2} = \frac{du^2 + dv^2}{v^2}$$

Example 1. *Consider the map $T : \mathbb{H}^2 \to \mathbb{H}^2$ defined by*

$$T(x, y) = (x + a, y)$$

where a is a real number. In this case, the map shifts x-coordinate by a constant a, while leaving y-coordinate unchanged. So, we have

$$u(x, y) = x + a \quad \text{and} \quad v(x, y) = y$$

For this transformation, the differentials $du = dx$ and $dv = dy$, so

$$\frac{du^2 + dv^2}{v^2} = \frac{dx^2 + dy^2}{y^2}$$

Hence, this transformation T preserves the hyperbolic metric, and we conclude that T is an isometry.

Example 2. *Consider the transformation $T(x, y)$ defined by*

$$T(x, y) = (\lambda x, \lambda y)$$

where λ is a positive real number. Under this transformation, the component functions become

$$u(x, y) = \lambda x \text{ and } \quad v(x, y) = \lambda y$$

Differentiating, we get:

$$du = \lambda dx \quad \text{and } dv = \lambda dy$$

Substituting into the hyperbolic metric expression, we get:

$$\frac{du^2 + dv^2}{v^2} = \frac{\lambda^2(dx^2 + dy^2)}{\lambda^2 y^2} = \frac{dx^2 + dy^2}{y^2}$$

Thus, T preserves the hyperbolic distance and the transformation T is an ***isometry*** *of the upper-half plane.*

The map $T(x, y) = (x + a, y)$ performs horizontal translation and shifts each point along x-axis and the map $T(x, y) = (\lambda x, \lambda y)$ scales both coordinates by a positive scalar λ, radially away from or towards the origin. Both transformations leave the hyperbolic line element unchanged and are thus examples of **isometries** in the upper half-plane model.

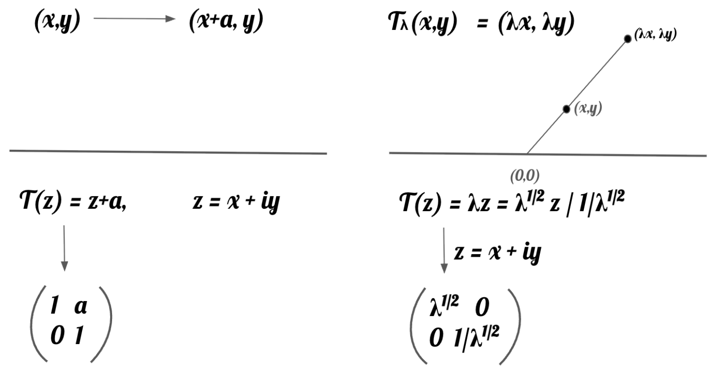

FIGURE 20. Transformation

Action of $\mathrm{SL}(2, \mathbb{R})$ **on** $\mathbb{H}^2$ **:** Examine the action of $\mathrm{SL}(2, \mathbb{R})$ on the upper half-plane, $\mathbb{H}^2$. The $\mathrm{SL}(2, \mathbb{R})$ consists of all 2×2 real matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad ad - bc = 1,$$

where $a, b, c, d \in \mathbb{R}$.

$$\mathrm{SL}(2, \mathbb{R}) \times \mathbb{H}^2 \to \mathbb{H}^2$$
$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) \to \frac{az + b}{cz + d}$$

If $z \in \mathbb{H}^2$, then $T(z) = \frac{az+b}{cz+d} \in \mathbb{H}^2$, where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. The imaginary part of $T(z)$ can be expressed as:

$$\mathrm{Im}(T(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2} > 0,$$

which confirms that $T(z) \in \mathbb{H}^2$. The group $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathbb{H}^2$ through Möbius transformations. So, T is a *homeomorphism* and establishes a continuous one-to-one mapping from $\mathbb{H}^2$ to itself.

Action of $\mathrm{PSL}(2, \mathbb{R})$ **on** $\mathbb{H}^2$ **:**

Definition 5.2. *Let* $\mathbb{H}^2 = \{z \in \mathbb{C} \mid Im(z) > 0\}$ *denote the upper half plane. The group* $\mathrm{PSL}(2, \mathbb{R})$ *is defined as the quotient group:*

$$\mathrm{PSL}(2, \mathbb{R}) := \mathrm{SL}(2, \mathbb{R})/\{\pm I\}$$

where $\mathrm{SL}(2,\mathbb{R})$ *is the group of all* 2×2 *real matrices with determinant 1, and* I *is the identity matrix. We define a map*

$$\varphi : \mathrm{PSL}(2,\mathbb{R}) \to Isom^{+}(\mathbb{H}^2)$$

by letting φ *act on* $z \in \mathbb{H}^2$ *as a Möbius transformation:*

$$\varphi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) = \frac{az+b}{cz+d}.$$

Theorem 5.1. *The map* φ *defines an injective group homeomorphism from* $\mathrm{PSL}(2,\mathbb{R})$ *into the group of orientation-preserving isometries of* $\mathbb{H}^2$

Proof. Let $z = x + iy \in \mathbb{H}^2$, with y > 0. Consider the Möbius transformation:

$$T(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d) \in \mathbb{R}$ and $ad - bc = 1$.

We first verify that $T(z) \in \mathbb{H}^2$. The imaginary part of $T(z)$:

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

Since $\mathrm{Im}(z) > 0$ and $|cz+d|^2 > 0$, it follows that $\mathrm{Im}(T(z) > 0$, so the image remains in $\mathbb{H}^2$. Now, we show that T preserves the hyperbolic metric. let $\gamma(t) =$ x(t) + iy(t) for $t \in [a, b]$ be a piecewise differentiable path in $\mathbb{H}^2$. The hyperbolic length of γ is:

$$L(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt.$$

The complex derivative of T is:

$$T'(z) = \frac{1}{(cz+d)^2}.$$

The imaginary part transforms as:

$$\mathrm{Im}(T(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

So,

$$|T'(z)| = \frac{\mathrm{Im}(T(z))}{\mathrm{Im}(z)}.$$

Therefore, for $z = \gamma(t)$, we find:

$$|T'(\gamma(t))| \cdot |\gamma'(t)| = \frac{\mathrm{Im}(T(\gamma(t)))}{\mathrm{Im}(\gamma(t))} \cdot |\gamma'(t)|.$$

Thus, the length of the image path is:

$$L(T \circ \gamma) = \int_a^b \frac{|T'(\gamma(t))| \cdot |\gamma'(t)|}{\mathrm{Im}(T(\gamma(t)))}\, dt = \int_a^b \frac{|\gamma'(t)|}{\mathrm{Im}(\gamma(t))}\, dt = L(\gamma).$$

Hence, T preserves hyperbolic length and is an isometry of $\mathbb{H}^2$.

Because Möbius transformations with real coefficients and determinant one are known to be orientation-preserving homeomorphisms of $\mathbb{H}^2$, and since the kernel of the map $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{Isom}^{+}(\mathbb{H}^2)$ is $\{\pm I\}$, the map φ is injective. Finally, since matrix multiplication corresponds to function composition of Möbius maps, φ is a group homeomorphism. Thus, $\mathrm{PSL}(2,\mathbb{R})$ embeds as a subgroup of $\mathrm{Isom}^{+}(\mathbb{H}^2)$.

Let us take:

$$\mathrm{SL}(2,\mathbb{R}) \to \mathrm{Isom}^{+}(\mathbb{H}^2)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az+b}{cz+d} \right)$$

This transformation is a group homeomorphism, and let's denote this map as ψ.

$$\psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = I_d \quad I_d(z) = z$$

$$\frac{az+b}{cz+d} = z$$

$$az + b = cz^2 + dz$$

$$cz^2 + (d-a)z - b = 0 \quad \forall\ z \in\ \mathbb{H}^2)$$

This implies that the coefficients must vanish. Thus, we obtain $c = b = 0$, $d = a$. Also, $ad - bc = 1$ implies $a = \pm 1$ and $d = \pm 1$. So,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} or \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence, ker ψ is the subgroup consisting of these two matrices, $\pm I_d$. □

Conclusion. We define the quotient map from PSL$(2, \mathbb{R})$, as ψ, alongside another map φ. Notably, these two maps commuteand on composing them, we find that $q = \psi$.

So, we can assert that φ is, in fact, a monomorphism—an injective homeomorphism. This injectivity implies that PSL$(2, \mathbb{R})$ is isomorphic to a subgroup of the isometry group of the upper half-plane, Isom$^+(\mathbb{H}^2)$.

6. Exploring Mobius Transformations and the Geodesics of the Upper Half-Plane

Definition 6.1. *A* Möbius transformation *is a mapping, $T : \mathbb{C} \to \mathbb{C}$ of the form,*

$$T(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$, and $ad - bc \neq 0$. The condition $ad - bc \neq 0$ ensures that the transformation is non-degenerate(i.e. invertible).

Proof. Let $S(z) = \dfrac{dz-b}{-cz+a}$. S is a Möbius transformation since $ad - bc \neq 0$ holds.

$$T \circ S(z) = I_d(z), \quad I_d(z) = z$$

and

$$S \circ T = I_d$$

Therefore, S is inverse of T. T is a bijection. If S, T are Möbius transformations then $S \circ T$ is again a Möbius transformation. This proves that the set of all Möbius transformations forms a group under the operation of composition.

This definition of Möbius transformations can be extended to $\mathbb{C} \cup \{\infty\}$, where $\mathbb{C} \cup\ \infty$ represents the one-point compactification of the complex plane $\mathbb{C}$.

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\} \quad \text{one-point compactification of } \mathbb{C}$$

The open sets in this one-point compactification are the open sets of $\mathbb{C}$, along with the set $V = (\mathbb{C} \setminus K) \cup \infty$, where K is a closed and compact subset of $\mathbb{C}$. These are the open sets in $\mathbb{C}_\infty$. We can extend the definition of a Möbius transformation to $\mathbb{C}_\infty$, or $\mathbb{C} \cup \infty$. For a Möbius transformation T, we have:

$$T(z) = \frac{az+b}{cz+d}$$

For $z \in \mathbb{C}$, where a, b, c, d are complex numbers, and $ad - bc \neq 0$. Additionally, the transformation is defined as:

$$T(\infty) = \frac{a}{c}$$
$$T : \mathbb{C}_\infty \mapsto \mathbb{C}_\infty$$
$$T\left(\frac{-d}{c}\right) = \infty$$

□

Example 3. ***Translation:***

$$T(z) = z + a$$

where $a \in \mathbb{C}$

Example 4. ***Dilation:***

$$T_\lambda(z) = \lambda z$$

where $\lambda \neq 0$

Example 5. ***Rotation:***

$$R_\theta(z) = e^{i\theta} z$$

where θ *is a real number, giving a rotation about the origin.*

Example 6. ***Inversion:***

$$I(z) = \frac{1}{z}$$

which reflects the point across the unit circle.

Proposition 6.1. *A Möbius transformation can be expressed as a composition of translations, dilations, and inversions.*

Proof. Let $S(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$.
Case 1: Suppose $c = 0$
In this case, the transformation simplifies to:

$$S(z) = \frac{a}{d} z + \frac{b}{d}$$

Define

$$S_2(z) = \frac{a}{d} z, \quad S_1(z) = z + \frac{b}{d}$$
$$S(z) = S_2 \circ S_1(z)$$

Case 2: Case 2: Suppose $c \neq 0$ In this case, we perform the following steps: First, we apply the translation, then inversion, then dilation followed by another translation:

$$S_1(z) = z + \frac{d}{c}, \quad S_2 = \frac{1}{z}$$
$$S_3(z) = \frac{bc - ad}{c^2} z, \quad S_4(z) = z + \frac{a}{c}$$

Thus, the original transformation $S(z)$ can be written as:

$$S(z) = S_4 \circ S_3 \circ S_2 \circ S_1$$

□

7. Applications of Hyperbolic Geometry

Hyperbolic geometry has diverse applications in the field of mathematics, and multiple modern scientific domains. In Special Relativity, the spacetime geometry can be interpreted using hyperbolic models. Hyperbolic geometry has greatly influenced art and architecture, through its distinct aesthetic properties and visually striking nature of hyperbolic forms. The Poincaré Disk Model provides a framework for hyperbolic tessellations, forming visually appealing and intricate patterns. Modern architects employ hyperbolic geometry to create visually striking buildings such as The Gherkin in London, and the Beijing National Aquatic Center highlighting that hyperbolic designs optimize form and function in construction.

Hyperbolic geometry has become a powerful tool in computer science. It is used to analyze and visualize complex networks and data structures. It has enhanced machine learning tasks, thereby providing high-dimensional data while preserving proximity relationships and is valuable in graph-based algorithms, clustering etc., where Euclid methods fail.

Hyperbolic geometry and number theory concepts together have led to an improvement in cryptography and algorithmic designs. Mathematical properties of hyperbolic geometry achieve secure encryption and decryption in elliptic curve cryptography (ECC) processes. These connections demonstrate how abstract geometric ideas can drive practical advancement in cybersecurity and computational number theory. In theoretical physics, it serves

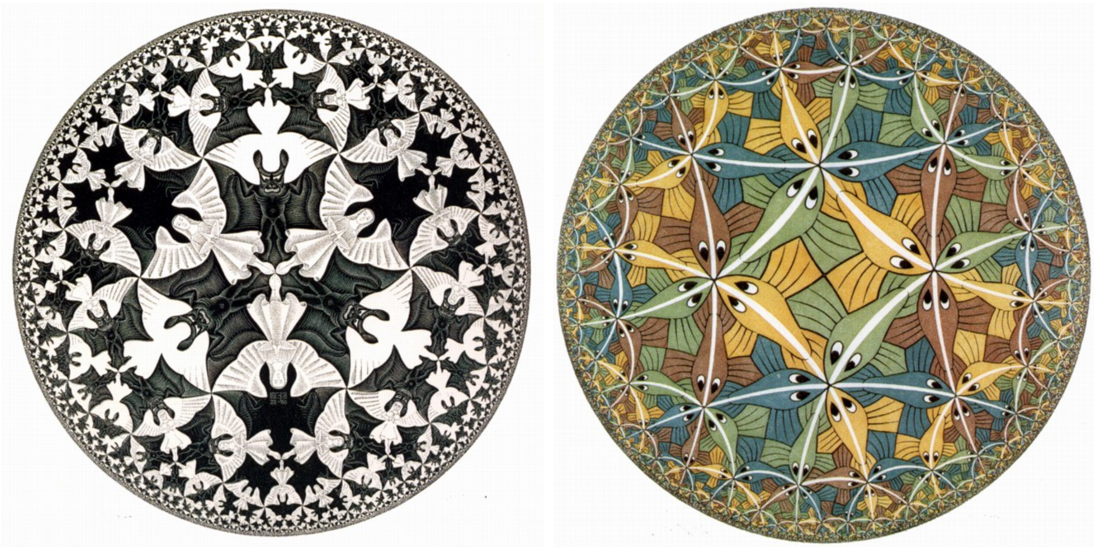

FIGURE 21. Tessellations by M.C. Escher, (a) Circle Limit IV (Heaven and Hell), 1960. (b) Circle Limit III, 1959.

as a foundational network for studying curvature, symmetry, and geometric transformations. Hyperbolic geometry is also used to investigate minimal surfaces, curvature flows, and variational problems. It also excels in educational field making abstract mathematical concepts tangible through interactive tools. By integrating hyperbolic geometry into curricula, educators foster deeper spatial reasoning, preparing students to tackle modern challenges in data science, physics and beyond.

8. Conclusion

Hyperbolic geometry, a non-Euclidean Geometry arises by relaxing either the metric requirements or replacing the parallel postulate with an alternative. Its dismissal of the parallel postulate leads to a consistent geometric system in which infinite lines pass through a point and never intersect a given line.

With the help of rigorous models such as the Poincaré disk, the hyperboloid model, the Klein model and the Poincaré half plane model, we understood not only the behaviour of the negatively curved surfaces but could also validate that Hyperbolic geometry exists.

We could understand the metrics, geodesics, hyperbolic distance and isometry in each of the models. Beyond these theoretical implications, hyperbolic geometry plays an integral role in real-life applications in cosmology, physics, neural networks, and biology. It even finds expression in the intricate art by M.C Escher where symmetry and tessellations reflect hyperbolic space.

In conclusion, hyperbolic geometry extends the boundaries of spatial understanding. It challenges the old assumptions space, angle and parallelism while offering tools for innovation in science, technology, and arts.

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