PLATEAU'S PROBLEM

RYAN BANSAL

1. Introduction

Soap films spanning wire frames pull tight and settle into shapes that look as if they are trying to use as little surface area as possible. Joseph Plateau studied this systematically and distilled a simple guiding rule: at equilibrium the film minimizes area. In this paper we take that rule and translate it to differential geometry to ask a simple existence question: does there always exist a surface that spans a given loop and minimizes area?

In the 1930s, Jesse Douglas and Tibor Radó solved this classical problem by a variational method. For his work, Douglas received the 1936 Fields Medal. Later developments in geometric measure theory broadened the picture and gave more results, but here we follow the original parametric disc approach and retrace the Douglas–Radó existence theory.

Given a rectifiable Jordan curve $\Gamma \subset \mathbb{R}^N$, we seek a disc-shaped surface that spans Γ and has the least possible surface area among all such discs. Minimizing area directly often misbehaves under limits, so we instead work with an energy that records how much a map from the unit disc stretches. This energy is quadratic, behaves well under limits, and decreases under natural relaxation while the boundary stays fixed. Starting from any spanning disc, we relax it without moving the boundary and obtain a sequence that settles to a limit surface. The limit balances its local stretch in perpendicular directions; in that balanced regime, area equals energy, so the limit is also area-minimizing and is smooth in the interior. In this paper we will work in $\mathbb{R}^N$ with $N \geq 3$, focusing on surfaces which are homeomorphic to the disc.

Exact formulas for minimal discs are rare, so after proving existence we show how to approximate the surface by leveraging mean curvature flow. Mean curvature points the way, and mean curvature flow is the descent that follows it. We will work with graphs over the unit disc D with a prescribed boundary value φ to find algorithms which estimate the minimal surface.

2. Setup

We fix a simple domain and parametrize the surface as a map. We take the unit disc

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}, \qquad \partial D = S^1,$$

and a map $X\colon D \to \mathbb{R}^N$. The boundary ∂D, or equivalently the circle S^1, will be sent to the given curve Γ. We can think of X as a soap film stretched across Γ. Different parametrizations can describe the same geometric surface, so the quantities we use should depend only on the geometry induced by X. For our equations we write $X_x = \partial X/\partial x$ and $X_y = \partial X/\partial y$ for short.

The quantity we ultimately care about is area.

Date: August 17, 2025.

Definition 2.1 (Area Functional)**.** The area of a map $X\colon D \to \mathbb{R}^N$ is

$$\mathrm{Area}(X) = \int_D |\mathrm{Jac}\, X| \int_D \sqrt{|X_x|^2|X_y|^2 - (X_x \cdot X_y)^2}\, dx\, dy.$$

In $\mathbb{R}^3$ this simplifies to $\mathrm{Area}(X) = \int_D |X_x \times X_y|\, dx\, dy$.

However, trying to minimize area directly is hard because the functional is not stable under limits. A more stable functional is the average squared stretch, which we call the *energy.*

Definition 2.2 (Dirichlet energy)**.** The Dirichlet energy of X is

$$E(X) = \frac{1}{2} \int_D \left(|X_x|^2 + |X_y|^2\right) dx\, dy.$$

You can think of $E(X)$ as recording how much the map stretches on average. It is quadratic, monotone under natural relaxations of the surface that keep the boundary fixed, and it is stable under weak limits.

Throughout we work with maps $X \in W^{1,2}(D, \mathbb{R}^N) \cap C^0(\overline{D})$, so the boundary $X|_{\partial D}$ is well defined. Here $W^{1,2}(D, \mathbb{R}^N)$ means that X and its first partial derivatives are square–integrable on D, and $C^0(\overline{D})$ means X extends continuously to the closed disc. This ensures $X|_{\partial D}$ is a well-defined continuous boundary map. When we say the boundary map runs once around Γ, we mean it parametrizes Γ weakly monotonically of degree one.

Definition 2.3 (Equicontinuity)**.** A family $\{f_\alpha\}$ on S^1 is equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f_\alpha(e^{i\theta}) - f_\alpha(e^{i\varphi})| < \varepsilon$ whenever $|\theta - \varphi| < \delta$, for all indices α.

Equicontinuity essentially means a single small angular change on the circle controls all maps in a family. It gives a common modulus of continuity and is exactly what is needed to apply Arzelà–Ascoli on the boundary.

Definition 2.4 (Weakly monotone boundary map)**.** A continuous map $\gamma\colon S^1 \to \Gamma$ is weakly monotone of degree one if it is a uniform limit of homeomorphisms of S^1 onto Γ.

Weakly monotone of degree one means that we traverse Γ once without backtracking. The parametrization may pause on short arcs, but it never reverses direction or skips around. Both definitions protect compactness of the boundary in minimizing sequences and make the "runs once around Γ" condition robust under uniform limits.

The way a parametrization can "waste" area is by stretching much more in one direction than the other, or by letting the coordinate lines fail to meet at right angles. The balanced case is when the two principal stretches match and the coordinate directions are orthogonal.

Definition 2.5 (Conformality)**.** A map is conformal at a point if $X_x \cdot X_y = 0$ and $|X_x| = |X_y|$ there. It is almost conformal if these equalities hold almost everywhere.

Through this definition, we can make area and energy talk to each other. We first record the basic inequality relating them.

Theorem 2.6 (Energy dominates area)**.** *For any map $X\colon D \to \mathbb{R}^N$, we have $\mathrm{Area}(X) \le E(X)$, with equality if and only if X is conformal almost everywhere.*

Proof. By Cauchy–Schwarz, $|X_x \cdot X_y| \le |X_x|\, |X_y|$, hence

$$\sqrt{|X_x|^2|X_y|^2 - (X_x \cdot X_y)^2} \;\le\; |X_x|\, |X_y|.$$

By the AM-GM inequality,

$$|X_x|\,|X_y| \;\le\; \tfrac{1}{2}\big(|X_x|^2 + |X_y|^2\big).$$

Combining these gives the pointwise bound "area density $\le$ energy density", and then integrating over D yields $\mathrm{Area}(X) \le E(X)$. Equality holds exactly when $X_x \cdot X_y = 0$ and $|X_x| = |X_y|$ almost everywhere, i.e., when X is conformal. ■

Area measures geometric size; energy measures squared stretch. Energy always dominates area and becomes large mainly when stretch is uneven. By gently reparametrizing the disc (keeping the rim fixed), we can even out that unevenness so a map's energy nearly tracks its true area. This is why we will minimize energy: once the minimizer is (almost) conformal, Theorem 2.6 turns the inequality into equality and identifies an area minimizer.

To find the least stretching maps, which minimize energy, we will use harmonic maps.

Definition 2.7 (Harmonic map)**.** A map $X\colon D \to \mathbb{R}^N$ is harmonic in D if each component satisfies the Laplace equation $\Delta X = X_{xx} + X_{yy} = 0$ in D.

Going back to the rubber sheet picture, harmonic means there is no net pull anywhere in the interior. The tension is evenly distributed for the given boundary. Equivalently, each component has the property that its value at a point equals the average over small surrounding circles. This is the two–dimensional analogue of a stretched rubber sheet settling into equilibrium. If energy is our target, the extremals are harmonic maps, so we aim to understand them next.

Theorem 2.8 (Energy minimality of harmonic maps)**.** *Let H be harmonic in D with prescribed boundary values $H|_{\partial D} = g$. If X has the same boundary values $X|_{\partial D} = g$, then*

$$E(X) = E(H) + \frac{1}{2}\int_D \big(|(X-H)_x|^2 + |(X-H)_y|^2\big)\,dx\,dy \;\ge\; E(H),$$

with equality if and only if $X \equiv H$.

Proof. Write $Y = X - H$, so $Y|_{\partial D} = 0$. Expanding the square gives

$$2E(X) = \int_D \big(|H_x+Y_x|^2+|H_y+Y_y|^2\big) = 2E(H)+2\int_D \big(H_x\cdot Y_x+H_y\cdot Y_y\big)+\int_D \big(|Y_x|^2+|Y_y|^2\big).$$

Since $Y \in W_0^{1,2}(D,\mathbb{R}^N)$, integration by parts is justified componentwise, and using $Y|_{\partial D} = 0$ yields

$$\int_D \big(H_x\cdot Y_x + H_y\cdot Y_y\big) = -\int_D (H_{xx}+H_{yy})\cdot Y = -\int_D (\Delta H)\cdot Y = 0,$$

since H is harmonic. Therefore

$$E(X) = E(H) + \tfrac{1}{2}\int_D \big(|Y_x|^2 + |Y_y|^2\big)\,dx\,dy \;\ge\; E(H),$$

with equality exactly when $Y \equiv 0$, i.e., $X \equiv H$. ■

We will also use that working with energy is equivalent to working with area at the level of infima in our admissible class. Intuitively: with three pins on the rim, we can gently "re-time" the domain to make any competitor's energy as close to its geometric area as we like. The next lemma makes this precise and closes the gap.

Lemma 2.9 (Infimum of area equals infimum of energy). *In the admissible class $\mathcal{E}_\Gamma$ (with boundary parametrization free up to the three pinned points),*

$$\inf_{X\in\mathcal{E}_\Gamma} \operatorname{Area}(X) \;=\; \inf_{X\in\mathcal{E}_\Gamma} E(X).$$

Proof. The inequality $\inf \operatorname{Area} \le \inf E$ follows from Theorem 2.6. For the reverse inequality, fix any competitor f and any $\varepsilon > 0$. For $\delta > 0$ define the lifted map $F_\delta(x,y) = (f(x,y),\, \delta x,\, \delta y)$ into $\mathbb{R}^{N+2}$. For every $\delta > 0$ the differential DF_δ has full rank (the δ-directions prevent degeneracy), so the image surface $F_\delta(D)$ carries a continuous induced metric and hence admits local isothermal coordinates. On the disk these globalize to a conformal diffeomorphism $G_\delta \colon D \to F_\delta(D)$. Let g_δ be the projection of G_δ to $\mathbb{R}^N$, and let h_δ be the harmonic map with boundary values $g_\delta|_{\partial D}$. Then

$$E(h_\delta) \;\le\; E(g_\delta) \;=\; E(G_\delta) \;=\; \operatorname{Area}(G_\delta) \;=\; \operatorname{Area}(F_\delta),$$

using harmonic minimality for fixed trace, conformal invariance of energy, and equality of area and energy for conformal parametrizations. After projection $g_\delta = \pi \circ G_\delta$, the boundary trace still runs once around Γ (weakly monotone), and by precomposing with a disc automorphism we can enforce the three-point pinning without changing the represented surface. As $\delta \to 0$, the area integrand for F_δ is

$$\sqrt{|f_x|^2|f_y|^2 - (f_x\!\cdot\! f_y)^2 + \delta^2(|f_x|^2 + |f_y|^2) + \delta^4} \;\to\; \sqrt{|f_x|^2|f_y|^2 - (f_x\!\cdot\! f_y)^2},$$

and it is dominated by $\frac{1}{2}(|f_x|^2 + |f_y|^2) + 1$ on D, so dominated convergence applies and $\operatorname{Area}(F_\delta) \to \operatorname{Area}(f)$. Hence for small δ we get $E(h_\delta) \le \operatorname{Area}(f) + \varepsilon$. Taking the infimum over f and letting $\varepsilon \downarrow 0$ yields $\inf E \le \inf \operatorname{Area}$. ■

3. Boundary data and competitors

To model a film spanning a fixed loop $\Gamma \subset \mathbb{R}^N$, we ask for disc maps whose boundary trace runs once around Γ without backtracking.

Definition 3.1 (Weakly monotone boundary map). A continuous map $\gamma\colon S^1 \to \Gamma$ is weakly monotone of degree one if it is a uniform limit of homeomorphisms of S^1 onto Γ (equivalently, it may pause but not reverse direction on boundary arcs).

This lets the boundary slow down on short arcs without creating artificial backtracking; the degree-one condition encodes that we go once around Γ in the right orientation.

Definition 3.2 (Admissible class). Let $\mathcal{E}_\Gamma$ be the set of maps $X \in W^{1,2}(D, \mathbb{R}^N) \cap C^0(\overline{D})$ whose boundary trace $X|_{\partial D}$ parametrizes Γ weakly monotonically.

This class is flexible enough to contain all natural competitors we build and rigid enough to stay controlled. Traces exist for $W^{1,2}$ maps, and continuity up to $\overline{D}$ makes the spanning condition literal.

Boundary reparametrizations create a symmetry we want to remove. If we leave it in, a minimizing sequence can whirl around the boundary faster and faster while describing the same surface, breaking equicontinuity and compactness. Pinning three boundary points kills that symmetry without changing which surfaces are represented.

Definition 3.3 (Disc automorphisms). The conformal self–maps of D form a group under composition where each extends continuously to ∂D and is a homeomorphism of the circle.

The group acts triply transitively on ∂D (any ordered triple is sent to any other). Precomposition preserves $W^{1,2}(D,\mathbb{R}^N)$, continuity up to $\overline{D}$, and weakly monotone traces, so normalization does not change $\mathcal{E}_\Gamma$.

Lemma 3.4 (Three-point normalization)**.** *Fix distinct $\zeta_1,\zeta_2,\zeta_3 \in S^1$ and an ordered triple on Γ. By precomposing any $X \in \mathcal{E}_\Gamma$ with a disc automorphism, we may arrange $X(\zeta_i)$ equals the prescribed boundary points. This quotients the boundary Möbius symmetry and stabilizes the class.*

Proof. Since $X|_{\partial D}$ is weakly monotone of degree one onto Γ, each prescribed boundary point of Γ has a nonempty set of preimages on S^1, and the three preimages can be chosen in the correct cyclic order. Automorphisms of D act triply transitively on S^1 and extend to homeomorphisms of $\overline{D}$, so there is a unique automorphism sending those three preimages to ζ_1,ζ_2,ζ_3. Precomposition preserves $W^{1,2}$, continuity up to the boundary, and weak monotonicity of the trace, so the class is unchanged while the pins are enforced. ■

4. Existence via energy minimization

We now put the ingredients together. For a fixed boundary trace, replacing a competitor by its harmonic extension only lowers the energy. After the three-point pinning, harmonic maps with uniformly bounded energy have equicontinuous boundary traces. Together with weak lower semicontinuity of the Dirichlet energy, these facts let us pass to the limit in a minimizing sequence and produce a minimizer. The boundary modulus of continuity makes sure the boundary condition survives the limit via Arzelà–Ascoli.

We begin with a classical principle for fixed boundary data.

Theorem 4.1 (Dirichlet principle)**.** *If g is the boundary trace of some $W^{1,2}$ map, there exists a unique harmonic $H_g \in W^{1,2}(D,\mathbb{R}^N)\cap C^0(\overline{D})$ with $H_g|_{\partial D} = g$ minimizing E among all maps with trace g.*

Proof. Solve the scalar Dirichlet problem for each component by the Poisson formula to obtain the harmonic extension $H_g \in C^0(\overline{D})$ with $\Delta H_g = 0$ and $H_g|_{\partial D} = g$. For any competitor X with the same trace, write $X = H_g + Y$ with $Y \in W_0^{1,2}(D,\mathbb{R}^N)$. As in Theorem 2.8, the cross term vanishes, giving

$$E(X) = E(H_g) + \tfrac{1}{2}\int_D \left(|Y_x|^2 + |Y_y|^2\right) dx\,dy \geq E(H_g).$$

Thus H_g minimizes E. If H_1, H_2 are two minimizers with the same trace, the same identity with $Y = H_1 - H_2 \in W_0^{1,2}$ forces $\nabla Y \equiv 0$, hence $H_1 \equiv H_2$. ■

This gives the harmonic replacement step.

Proposition 4.2 (Harmonic replacement)**.** *For $X \in \mathcal{E}_\Gamma$, let H be the harmonic map with $H|_{\partial D} = X|_{\partial D}$. Then $H \in \mathcal{E}_\Gamma$ and $E(H) \leq E(X)$.*

Proof. The boundary trace and weak monotonicity are preserved by construction, so $H \in \mathcal{E}_\Gamma$. The energy inequality is exactly Theorem 2.8 with $g = X|_{\partial D}$. By the Dirichlet principle (Theorem 4.1), H is the unique minimizer of E among maps with the same trace, so replacing X by H lowers energy without changing the boundary class. ■

Near the rim, bounded energy tames oscillation. The next lemma makes that control precise.

Lemma 4.3 (Courant–Lebesgue). *For harmonic maps with uniformly bounded energy and under the three-point normalization, boundary traces on S^1 have a uniform modulus of continuity (hence form an equicontinuous family).*

Proof. Write $H(re^{i\theta})$ in polar coordinates. The energy identity is

$$E(H) = \tfrac{1}{2}\int_0^1 \int_0^{2\pi} \Big(|\partial_r H|^2 + \tfrac{1}{r^2}|\partial_\theta H|^2\Big)\, r\, d\theta\, dr.$$

For a.e. $r \in (0,1)$ we have

$$\int_0^{2\pi} |\partial_\theta H(re^{i\theta})|^2\, d\theta \;\le\; 2\, r\, E(H) \;\le\; 2E(H).$$

Hence for $\theta, \varphi \in [0, 2\pi]$ and any such r,

$$|H(re^{i\theta}) - H(re^{i\varphi})| \le \int_\varphi^\theta |\partial_\tau H(re^{i\tau})|\, d\tau \le |\theta-\varphi|^{1/2}\Big(\int_0^{2\pi} |\partial_\theta H(re^{i\theta})|^2\, d\theta\Big)^{1/2} \le (2E(H))^{1/2}\, |\theta-\varphi|^{1/2}.$$

Next, for each fixed θ and $r < \rho < 1$,

$$|H(e^{i\theta}) - H(re^{i\theta})| \le \int_r^1 |\partial_\sigma H(\sigma e^{i\theta})|\, d\sigma \le \Big(\int_r^1 \sigma\, d\sigma\Big)^{1/2}\Big(\int_r^1 \tfrac{1}{\sigma}\, |\partial_\sigma H(\sigma e^{i\theta})|^2\, d\sigma\Big)^{1/2}.$$

Integrating in θ and using the energy identity yields

$$\sup_\theta |H(e^{i\theta}) - H(re^{i\theta})| \le (1-r)^{1/2}\, (2E(H))^{1/2}.$$

Combining the two displays gives, for $\delta \in (0, \pi]$, the compact form

$$|H(e^{i\theta}) - H(e^{i\varphi})| \le C\, E(H)^{1/2}\big(|\theta-\varphi|^{1/2} + (1-r)^{1/2}\big)$$

whenever $|\theta - \varphi| \le \delta$. Choosing $r = 1 - |\theta - \varphi|$ gives a uniform Hölder-1/2 modulus on ∂D depending only on $E(H)$. The three-point normalization prevents collapse of large arcs and yields uniform control along the whole boundary. By Arzelà–Ascoli, the boundary traces form a compact family in $C^0(S^1)$. This is the boundary equicontinuity estimate we need at ∂D. The three pins prevent long arcs on S^1 from collapsing to tiny arcs on Γ, so weak monotonicity is preserved in the boundary limit. ■

All the pieces are now in place to pass to the limit and get a minimizer.
Putting these ingredients together produces a genuine minimizer in our class.

Theorem 4.4 (Existence of an energy minimizer). *With three-point normalization, there exists $X_\infty \in \mathcal{E}_\Gamma$ that minimizes E over the three-point normalized subclass of $\mathcal{E}_\Gamma$.*

Proof. Pick a minimizing sequence $(X_k) \subset \mathcal{E}_\Gamma$. By Proposition 4.2, replace it with the harmonic sequence (H_k) with the same traces, so $E(H_k) \le E(X_k)$ and $H_k \in \mathcal{E}_\Gamma$. The energies are bounded, so Lemma 4.3 and the three pins give equicontinuity on ∂D, hence $H_k|_{\partial D} \to g_\infty$ uniformly. A uniform $W^{1,2}$ bound implies (after a subsequence) $H_k \rightharpoonup X_\infty$ weakly in $W^{1,2}(D, \mathbb{R}^N)$; standard interior estimates for harmonic maps give a uniform interior Hölder bound, hence local uniform convergence in D. Lower semicontinuity of E under weak $W^{1,2}$ convergence gives $E(X_\infty) \le \liminf_k E(H_k)$. The boundary limit is weakly monotone and respects the pins, so $X_\infty \in \mathcal{E}_\Gamma$ and is a minimizer. ■

5. Conformality and minimal surface

Finally, we link energy minimality to area minimality.

An energy minimizer could still waste area by skewing one coordinate direction against the other. The only freedom left is to reparametrize the disk. Asking that energy not drop under those reparametrizations forces balance.

Proposition 5.1 (Domain reparametrizations forces conformality)**.** *Let X_∞ be an energy minimizer in the normalized class. Considering variations by precomposition with disc automorphisms that fix the three pinned boundary points, the first variation of E forces X_∞ to be almost conformal a.e.*

Proof. First, X_∞ is harmonic in D: interior variations with compact support do not change the boundary class, so by Theorem 2.8 the Euler–Lagrange equation is $\Delta X_\infty = 0$.

For harmonic maps, the Hopf differential

$$\Phi = (\partial_z X_\infty)\cdot(\partial_z X_\infty)\, dz^2 \quad (\partial_z = \tfrac{1}{2}(\partial_x - i\,\partial_y))$$

is holomorphic. Consider any C^1 one-parameter family of disc automorphisms $\{\phi_t\}$ with $\phi_0 = \mathrm{id}$ and whose boundary values fix the three pinned points; its generating vector field $\xi = \partial_t \phi_t|_{t=0}$ is holomorphic in D and tangent to ∂D, vanishing at the pins. Minimality of E over the three-point–normalized subclass of $\mathcal{E}_\Gamma$ gives

$$\frac{d}{dt}\Big|_{t=0} E(X_\infty \circ \phi_t) = 0.$$

The first-variation formula for domain variations (applied componentwise) yields

$$0 = -4\,\Re \int_D \Phi\, \partial_{\bar z}\xi\, dx\, dy \; + \; \int_{\partial D} B(X_\infty, \xi),$$

where B is a boundary term depending on ξ and the tangential/normal derivatives of X_∞. Because ξ is tangent to ∂D and vanishes at the three pinned points, the boundary term is zero. Since ξ is holomorphic, $\partial_{\bar z}\xi \equiv 0$, so the identity reduces to

$$\Re \int_D \Phi\, h\, dx\, dy = 0 \qquad \text{for all holomorphic } h \text{ vanishing at the pins.}$$

Since the integral vanishes for every holomorphic h and Φ is holomorphic, we must have $\Phi \equiv 0$. Therefore $\partial_z X_\infty \cdot \partial_z X_\infty \equiv 0$, which is equivalent to $\langle X_x, X_y\rangle = 0$ and $|X_x| = |X_y|$ a.e., i.e., conformality. ■

We now conclude the argument.

Lemma 5.2 (Area equals energy at the minimizer)**.** *For X_∞ as above,* $\mathrm{Area}(X_\infty) = E(X_\infty)$.

Proof. By Theorem 2.6, Area $\le E$ with equality exactly for conformal maps. Proposition 5.1 yields (almost) conformality of X_∞, hence $\mathrm{Area}(X_\infty) = E(X_\infty)$. ■

Harmonicity (from the replacement step) together with conformality implies that X_∞ solves the minimal surface equation and is smooth in the interior; any branch points are isolated. Thus X_∞ is a disc-type minimal surface spanning Γ.

In an ambient dimension of $N = 3$ it is shown that interior branch points do not occur. However, in higher dimensions branch points may occur, but they are still isolated. When Γ has additional regularity (e.g. $C^{k,\alpha}$), standard elliptic theory improves boundary regularity of the spanning surface accordingly.

6. Approximating minimal surfaces

Now that we know a minimizer exists, the natural next question is how to actually find it. Exact formulas are rare, so we move the surface in the direction that lowers area. Mean curvature tells us which way to go; mean curvature flow is the gradient descent for area. To keep things simple and consistent with our setup, we work with graphs over D and fix the boundary value φ.

Take a function $f\colon \overline{D} \to \mathbb{R}$ with $f|_{\partial D} = \varphi$ and write its graph $\Sigma_f = \{(x, y, f(x, y)) : (x, y) \in D\}$. Its area is

$$A(f) = \int_D \sqrt{1 + |\nabla f|^2}\, dxdy.$$

Vary f by $f_t = f + t\eta$ with η compactly supported in D (so the boundary stays put). A quick computation and an integration by parts (the boundary term vanishes by the support of η) give

$$\frac{d}{dt}\Big|_{t=0} A(f_t) = \int_D \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \cdot \nabla\eta\, dxdy = -\int_D \operatorname{div}\Big(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\Big)\, \eta\, dxdy.$$

Stationary graphs satisfy the minimal surface equation

$$\operatorname{div}\Big(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\Big) = 0,$$

and with the upward unit normal the scalar mean curvature is the left-hand side,

$$H(f) = \operatorname{div}\Big(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\Big).$$

If we now let the surface relax by prescribing

$$\partial_t f = \sqrt{1 + |\nabla f|^2}\, \operatorname{div}\Big(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\Big) \quad \text{in } D, \qquad f|_{\partial D} = \varphi,$$

then any minimal graph is stationary and along smooth solutions the area decreases according to

$$\frac{d}{dt} A(f_t) = -\int_D H(f_t)^2 \sqrt{1 + |\nabla f_t|^2}\, dxdy \ \leq 0,$$

In practice we use a simple relaxation step that turns each update into a linear boundary value problem. Given f_k, set

$$w_k(x) = \frac{1}{\sqrt{1 + |\nabla f_k(x)|^2}},$$

and if $|\nabla f_k| \leq M$ a.e., then $(1 + M^2)^{-1/2} \leq w_k \leq 1$, so $\nabla\cdot(w_k \nabla\cdot)$ is uniformly elliptic. Find f_{k+1} with the same boundary trace by solving

$$\nabla\cdot\big(w_k\, \nabla f_{k+1}\big) = 0 \quad \text{in } D, \qquad f_{k+1}|_{\partial D} = \varphi.$$

This keeps the nonlinear weight from the previous iterate, makes the next step linear, and pushes us toward $H = 0$. When $|\nabla f_k|$ is bounded, the coefficient w_k is bounded above and below, so the Dirichlet problem has a unique solution. Equivalently, f_{k+1} uniquely minimizes

$$J_k(u) = \int_D w_k\, |\nabla u|^2\, dxdy \quad \text{among } u \text{ with } u|_{\partial D} = \varphi,$$

so $J_k(f_{k+1}) \le J_k(f_k)$. Euler–Lagrange of J_k with fixed trace is $\nabla\cdot(w_k\nabla u) = 0$, and strict convexity holds since $w_k \ge (1+M^2)^{-1/2} > 0$. If $f_{k+1} = f_k$, then

$$\nabla\cdot\Big(\frac{\nabla f_*}{\sqrt{1+|\nabla f_*|^2}}\Big) = 0,$$

so the scheme naturally seeks a minimal graph. You can view this as a backward Euler time step for the flow with the nonlinearity lagged:

$$\frac{f_{k+1}-f_k}{\tau} = \sqrt{1+|\nabla f_k|^2}\ \mathrm{div}\Big(\frac{\nabla f_{k+1}}{\sqrt{1+|\nabla f_k|^2}}\Big),$$

and as $\tau \to \infty$ this reduces to $\nabla\cdot(w_k\nabla f_{k+1}) = 0$. A true backward Euler step (finite τ) yields a discrete area decrease; the $\tau \to \infty$ relaxation guarantees decrease of J_k.

If $\|\nabla f_k\|_{L^\infty} \le M$ for all k with M small enough, the update map $T : f \mapsto f_{k+1}$ is a contraction on $H_0^1(D)$, so $f_k \to f_\infty$ solving the minimal surface equation. On a mesh it is just a weighted Poisson solve each step; keep φ on the boundary and stop when successive iterates change little or when the residual of the minimal surface equation is small. Optional damping

$$f_{k+1} \leftarrow \theta f_{k+1} + (1-\theta) f_k \qquad (0 < \theta \le 1)$$

can help when slopes are larger. This gives a simple way to move from any spanning graph toward a minimal one while keeping the boundary fixed throughout.

7. Conclusion

We bridged area to a more workable functional (energy), pinned the rim to stop reparametrization drift, relaxed the interior by harmonic replacement to get an energy minimizer, and then used domain reparametrization to force even stretch (conformality), so area equals energy at the minimizer.

For computation, minimal graphs are steady states of mean curvature flow. Moving by mean curvature ($\partial_t X = -H\,\nu$, boundary fixed) decreases area. Applications such as geometry processing leverage mean-curvature-type flows to fair and smooth surfaces while keeping salient features. In imaging and vision, curvature regularization underlies denoising. In interfacial flows, capillarity drives motion by curvature with prescribed contact angles at solids, modeling droplet and film dynamics in microfluidics such as a soap bubble.