ISOMETRIES OF THE HYPERBOLIC PLANE

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1. Introduction

Hyperbolic geometry was one of the most important discoveries of the nineteenth century. Pioneers such as Nikolai Lobachevsky and János Bolyai developed hyperbolic geometry axiomatically, without providing concrete models. Later, mathematicians like Eugenio Beltrami and Felix Klein constructed explicit models of the hyperbolic plane, and Henri Poincaré introduced the upper half-plane model. The aim of this paper is to study the relationship between the isometries of the upper half-plane and Möbius transformations.

2. The upper half-plane model

We will introduce the upper half-plane

$$\mathbb{H} = \{v + wi \in \mathbb{C} : w > 0\}$$

as a convenient model for studying hyperbolic geometry. The key point is that the pseudosphere, which initially serves as an example of a surface with constant negative curvature, has the first fundamental form

$$\frac{dv^2 + dw^2}{w^2}.$$

If we extend this expression to the entire domain $w > 0$, we obtain a well-defined metric on the whole upper half-plane. In this way, the fundamental metric properties can be analyzed in $\mathbb{H}$ in the same way as on the pseudosphere.

In particular, the geodesics on the pseudosphere correspond to vertical lines and semicircles orthogonal to the axis $w = 0$ in the upper half-plane. Thus, $\mathbb{H}$ preserves the same geometric behavior as the pseudosphere, but in a simpler setting for study. We will refer to geodesics in $\mathbb{H}$ as *hyperbolic lines.*

Hyperbolic geometry is a type of non-Euclidean geometry in which Euclid's fifth postulate does not hold. We shall present an important property of hyperbolic geometry.

Proposition 2.1.

- Let p and q be points in $\mathbb{H}$. Then there is a unique hyperbolic line joining p and q.
- Let l be a hyperbolic line in $\mathbb{H}$ and p is a point not on l, there are infinitely many hyperbolic lines passing through p that do not intersect l.

Proof. For (i), we need to consider two cases.

- If $\mathrm{Re}(p) = \mathrm{Re}(q) = r$. In this case, the Euclidean l line given by $l = \{z \in \mathbb{C} \mid \mathrm{Re}(z) = r\}$ passes through p and q and is perpendicular to the real axis. So the line $l_h = l \cap \mathbb{H}$ is a hyperbolic line passing through p and q.

Date: August 18, 2025.

- If $\mathrm{Re}(p) \neq \mathrm{Re}(q)$ then the perpendicular bisector of the line segment joining p and q must intersect the real axis at a unique point r. So, the euclidean circle $\mathcal{C}$ with centre r and radius $|p-r| = |q-r|$ passes through p and q. The intersection $\mathcal{C} \cap \mathbb{H}$ is a hyperbolic line passing through p and q.

In each case the euclidean circles and lines that we used to construct this hyperbolic line are unique. So the hyperbolic line passing through p and q must be unique.

For (ii), we need to consider two cases.

- If l is a vertical line passing through $r \in \mathbb{R}$ then, $\mathrm{Re}(p) \neq r$. So there is a vertical line l' passing through p and $r' \in \mathbb{R}$. Then the Euclidean circle passing through p and q is parallel to l for any q between r and r'.
- If l is contained in a Euclidean semicircle $\mathcal{C}$, let $\mathcal{C}'$ be the Euclidean semicircle that is concentric to $\mathcal{C}$ and passes through p. Let q be a real number lying between C and C'. Then the semicircle D with center on $\mathbb{R}$ passing through p and q is disjoint from C and different from C', for every real q lying between C and C'.

■

3. Isometries of $\mathbb{H}$

Isometries are fundamental transformations in hyperbolic geometry because they preserve the first fundamental form and all metric properties. Studying these transformations allows us to understand the geometry of $\mathbb{H}$. In this section, we present some basic examples of isometries. Readers who wish to see a more geometric approach to isometries of the hyperbolic plane are referred to [Sti92].

Definition 3.1. A map $F : \mathbb{H} \to \mathbb{H}$ is called an *isometry* for this first fundamental form if, given

$$F(v, w) = (\tilde{v}(v, w), \tilde{w}(v, w)),$$

the first fundamental form is preserved

$$\frac{d\tilde{v}^2 + d\tilde{w}^2}{\tilde{w}^2} = \frac{dv^2 + dw^2}{w^2}$$

at every point of $\mathbb{H}$.

Some basic examples of isometries in $\mathbb{H}$ are the following.

Translation: $T_b(z) = z + b, \qquad b \in \mathbb{R},$

Positive dilation: $D_\lambda(z) = \lambda z, \qquad \lambda > 0,$

Inversion in the unit circle centered at the origin: $I(z) = -\dfrac{1}{z}.$

Proposition 3.2. Each of T_b, D_λ, and I is an isometry of $\mathbb{H}$.

Proof.
We will use the following fact.

Proposition 3.3. Let $\sigma(u, v)$ be a surface patch and let $\widetilde{\sigma}(\tilde{u}, \tilde{v})$ be a reparametrization of $\sigma(u, v)$ and let

$E\,du^2+2F\,du\,dv+G\,dv^2$, $\widetilde{E}\,d\tilde{u}^2+2\widetilde{F}\,d\tilde{u}\,d\tilde{v}+\widetilde{G}\,d\tilde{v}^2$ be their first fundamental forms. Then the differentials satisfy

$$du = \frac{\partial u}{\partial \tilde{u}}\,d\tilde{u} + \frac{\partial u}{\partial \tilde{v}}\,d\tilde{v}, \qquad dv = \frac{\partial v}{\partial \tilde{u}}\,d\tilde{u} + \frac{\partial v}{\partial \tilde{v}}\,d\tilde{v}.$$

1. Translation $T_b(z) = z + b$.

If $z = v + iw$ then $T_b(z) = \tilde{v} + i\tilde{w}$ with $\tilde{v} = v + b$, $\tilde{w} = w$. Hence

$$d\tilde{v} = dv, \qquad d\tilde{w} = dw,$$

and therefore

$$\frac{d\tilde{v}^2 + d\tilde{w}^2}{\tilde{w}^2} = \frac{dv^2 + dw^2}{w^2}.$$

2. Dilation $D_\lambda(z) = \lambda z$.

Write $z = v + iw$. Then $D_\lambda(z) = \tilde{v} + i\tilde{w}$ with $\tilde{v} = \lambda v$, $\tilde{w} = \lambda w$, so

$$d\tilde{v} = \lambda\,dv, \qquad d\tilde{w} = \lambda\,dw.$$

Hence

$$\frac{d\tilde{v}^2 + d\tilde{w}^2}{\tilde{w}^2} = \frac{\lambda^2(dv^2 + dw^2)}{(\lambda w)^2} = \frac{dv^2 + dw^2}{w^2}.$$

3. Inversion in the unit circle $I(z) = -1/z$.

Write $z = v + iw$. Then

$$I(z) = -\frac{1}{z} = -\frac{v - iw}{v^2 + w^2} = \tilde{v} + i\tilde{w},$$

so

$$\tilde{v} = -\frac{v}{v^2 + w^2}, \qquad \tilde{w} = \frac{w}{v^2 + w^2}.$$

By computing the partial derivatives:

$$\begin{aligned}
\frac{\partial \tilde{v}}{\partial v} &= \frac{-(v^2 + w^2) + 2v^2}{(v^2 + w^2)^2} = \frac{w^2 - v^2}{(v^2 + w^2)^2}, \\
\frac{\partial \tilde{v}}{\partial w} &= \frac{2vw}{(v^2 + w^2)^2}, \\
\frac{\partial \tilde{w}}{\partial v} &= \frac{-2vw}{(v^2 + w^2)^2}, \\
\frac{\partial \tilde{w}}{\partial w} &= \frac{v^2 - w^2}{(v^2 + w^2)^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
d\tilde{v} &= \frac{w^2 - v^2}{(v^2 + w^2)^2}\,dv + \frac{2vw}{(v^2 + w^2)^2}\,dw, \\
d\tilde{w} &= \frac{-2vw}{(v^2 + w^2)^2}\,dv + \frac{v^2 - w^2}{(v^2 + w^2)^2}\,dw.
\end{aligned}$$

And computing the numerator of the transformed form:

$$d\tilde{v}^2 + d\tilde{w}^2 = \frac{(w^2 - v^2)^2 + 4v^2w^2}{(v^2+w^2)^4}\,(dv^2 + dw^2)$$
$$= \frac{(v^2+w^2)^2}{(v^2+w^2)^4}\,(dv^2+dw^2) = \frac{1}{(v^2+w^2)^2}\,(dv^2+dw^2).$$

Since $\tilde{w} = \dfrac{w}{v^2+w^2}$ we have $\tilde{w}^2 = \dfrac{w^2}{(v^2+w^2)^2}$, hence

$$\frac{d\tilde{v}^2 + d\tilde{w}^2}{\tilde{w}^2} = \frac{\dfrac{1}{(v^2+w^2)^2}(dv^2+dw^2)}{\dfrac{w^2}{(v^2+w^2)^2}} = \frac{dv^2+dw^2}{w^2}.$$

This completes the proof. ■

3.1. **Möbius transformations.**

Möbius transformations are closely related to the isometries of $\mathbb{H}$. In this section, we study them and present the main result that every real Möbius transformation preserving $\mathbb{H}$ is an isometry. For a detailed treatment of Möbius transformations in the upper half-plane, the reader may consult [And05].

Definition 3.4. A *real Möobius transformation* is a map

$$M(z) = \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{R}, \quad ad - bc \neq 0.$$

Proposition 3.5. Let

$$M(z) = \frac{az+b}{cz+d},$$

be a real Möbius transformation. Then for $z = v + iw$ we have

$$\operatorname{Im} M(z) = \frac{(ad-bc)\,w}{|cz+d|^2}.$$

In particular, if M preserves $\mathbb{H}$ (maps $\mathbb{H}$ to itself), then $ad - bc > 0$.

Proof. Compute

$$M(z) = \frac{(av+b) + i\,aw}{(cv+d) + i\,cw} = \frac{\big((av+b)+i\,aw\big)\big((cv+d) - i\,cw\big)}{|cz+d|^2}$$
$$= \frac{(av+b)(cv+d) + aw\cdot cw + i\big(aw(cv+d) - (av+b)cw\big)}{|cz+d|^2}.$$

Hence

$$\operatorname{Im} M(z) = \frac{aw(cv+d) - cw(av+b)}{|cz+d|^2} = \frac{(ad-bc)\,w}{|cz+d|^2}.$$

For $w > 0$ the sign of $\operatorname{Im} M(z)$ equals the sign of $(ad-bc)$; if M preserves $\mathbb{H}$ then $\operatorname{Im} M(z) > 0$ for all $w > 0$, so $ad - bc > 0$. ■

Theorem 3.6. Every real Möbius transformation

$$M(z) = \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{R},$$

that preserves $\mathbb{H}$ is an isometry of $\mathbb{H}$.

Proof. Since M preserves $\mathbb{H}$, $ad - bc > 0$. If $c = 0$ then $M(z) = \dfrac{a}{d}z + \dfrac{b}{d}$. Thus, $M(z) = T_{b/d} \circ D_{a/d}(z)$ (note $a/d > 0$ since $ad - bc = ad > 0$).

If $c \neq 0$ a rearrangement gives

$$\begin{aligned} M(z) = \frac{az+b}{cz+d} &= \frac{a}{c} - \frac{ad-bc}{c^2} \cdot \frac{1}{z + \frac{d}{c}} \\ &= T_{a/c} \circ D_{(ad-bc)/c^2} \circ I \circ T_{d/c}(z), \end{aligned}$$

where $T_t(z) = z + t$, $D_\lambda(z) = \lambda z$, and $I(z) = -1/z$. Since $ad - bc > 0$, the factor $(ad-bc)/c^2$ is positive and each factor map is an isometry, and since any composition of isometries is an isometry, M is an isometry. ■

References

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[Sti92] John Stillwell. *Geometry of Surfaces*. Universitext. Springer, New York, 1992.

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