CALCULUS OF VARIATIONS

RISHI BALAJI

1. Introduction

1.1. **Overview.** Calculus of variations, also called variational calculus, is a field of mathematics that is used for finding the minima and maxima of *functionals* – which are "functions of functions" – by using small changes called *variations* in the functions.

Calculus of variations relates to differential geometry in that it is often used in geometric contexts. For instance, in many classic calculus of variations problems, the functionals that are being optimized consist of functions that are curves between two points or a shape with the given boundary.

Such calculus of variations problems include problems about geodesics, brachistochrones, and isoperimetric shapes, which this paper will go more in depth with.

1.2. **History.** Many notable mathematicians have been involved in the history of calculus of variations.

The field goes back to the work of Isaac Newton in the 17th century, who formulated and solved Newton's minimal resistance problem along with the brachistochrone problem presented by Johann Bernoulli, both using techniques from calculus of variations. After Newton's pioneering work, Leonhard Euler was one of the mathematicians who further developed the field in the early 18th century. Euler influenced Joseph-Louis Lagrange to contribute to the field, who then in turn influenced Euler to favor Lagrange's analytic approach to problems, and he then named the field *calculus of variations*.

Towards the beginning of the 19th century, mathematicians such as Carl Friedrich Gauss, Siméon Poisson, and Carl Jacobi made contributions to the field. Other significant work was improved by Augustin-Louis Cauchy, and Karl Weierstrass helped refine and establish the subject even further. More recently, in the 20th century, important contributers included David Hilbert, Emmy Noether, Henri Lebesgue, and Marston Morse, who applied calculus of variations to the field of Morse theory.

1.3. **Outline.** This paper provides a brief introduction to some of the fundamental ideas in calculus of variations, with a focus on applying those concepts to solve various problems. We begin with some necessary definitions and background that will be relevant throughout the paper. Then, we describe an important method for solving calculus of variations problems: the Euler-Lagrange equation. Finally, we give some examples of classic problems in calculus of variations and walk through their solutions in detail, explaining how to make use of the previously described methods and techniques.

Date: August 17, 2025.

2. Preliminaries

2.1. **Definitions.** Here we present some of the fundamental concepts that will be necessary to solve calculus of variations problems.

Definition 2.1 (Functional)**.** A *functional* is a function $J : S \to \mathbb{R}$ that maps a set of functions S to the real numbers $\mathbb{R}$. The value of such a functional can be denoted as $J[f]$, where f is a function in S.

Example. One example of a functional is the definite integral,

$$J[f] = \int_a^b f(x)\, dx.$$

Definition 2.2 (Stationary Point)**.** A *stationary point* of a functional is a function that does not change the value of the functional to the first order when a small change (a variation) is applied to the function. This concept is analogous to the stationary point of a regular function.

The *extrema* of a functional are stationary points, so finding these points is important for solving problems.

Definition 2.3 (Variation)**.** A function $f(x)$ can be changed to a new function $f(x)+\varepsilon\eta(x)$ for an arbitrary differentiable function $\eta(x)$ and small constant ε. The term $\varepsilon\eta(x)$ is called a *variation* of f and can be denoted as δf.

Definition 2.4 (Boundary Condition)**.** *Boundary conditions* are constraints for functions at the boundary of a functional's domain. They provide specific information about the values of a functional, which can be used to find unique and unambiguous solutions for the extremum of a functional

Definition 2.5 (Second Variation)**.** The *second variation* is a way to determine the nature of a stationary point (whether it is a maximum vs. a minimum, or a saddle point). It is analogous to the second derivative test in regular calculus.

2.2. **The Euler-Lagrange Equation.** The *Euler-Lagrange equation* is likely the most common method for solving calculus of variations problems.

It is a second-order ordinary differential equation whose solution provides the stationary points of the functional, which can be extrema functions.

Let J be a functional (called the *action functional*)

$$J[f] = \int_a^b L(x, f(x), f'(x))\, dx,$$

where a and b are constants, f is twice continuously differentiable, and the integrand L (called the *Lagrangian*) is twice continuously differentiable. The Lagrangian can also be defined using a *Lagrange multiplier* λ when dealing with constraints that involve multiple functionals, such as having $L = I[f] + \lambda J[f]$.

The Euler-Lagrange equation for J is

$$\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0.$$

A function f is a stationary point of J if and only if it satisfies the Euler-Lagrange equation.

In essence, the Euler-Lagrange equation can be derived by applying variations to a given function for a functional to check whether the functional extremizes the functional.

A special case of the Euler-Lagrange equation when L is not dependent on x is the *Beltrami identity*, which states that

$$L - y'\frac{\partial L}{\partial y'} = C$$

for some constant C.

3. Problems

3.1. Geodesic Problems.

Problem. What is the shortest curve between two points on the plane? This curve is the *geodesic* between the two points.

Solution. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two points on the plane. Let the function $y(x)$ represent a possible curve between the two points.

The quantity we would like to minimize is the arc length of y, so we can set up an arc length functional:

$$J[y] = \int_A^B ds = \int_A^B \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx.$$

In order to minimize J, we can find a stationary point using the Euler-Lagrange equation. In this case, the Lagrangian $L(x, y(x), y'(x)) = \sqrt{1 + (y')^2}$. So the corresponding Euler-Lagrange equation is

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0 - \frac{d}{dx}\frac{y'}{\sqrt{1 + (y')^2}} = 0.$$

We can now integrate both sides with respect to x to get that

$$-\frac{y'}{\sqrt{1 + (y')^2}} = C$$

for some constant C, and squaring this gives us

$$\begin{aligned}
\frac{(y')^2}{1 + (y')^2} &= C^2 \\
(y')^2 &= C^2 + C^2(y')^2 \\
(y')^2 - C^2(y')^2 &= C^2 \\
(y')^2 &= \frac{C^2}{1 - C^2} \\
y' &= \frac{C}{\sqrt{1 - C^2}}
\end{aligned}$$

Let $m = \frac{C}{\sqrt{1-C^2}}$, which is also a constant. So $y' = m$, and integrating with respect to x we have

$$y = mx + b$$

for some constant b. This is the slope-intercept form of a line, but we do not immediately know what the constants m and b are exactly.

In order to find m and b, we can use boundary conditions from the coordinates of points A and B. We know that $y(x_1) = y_1$ and $y(x_2) = y_2$, so we can produce two equations for m

and b:

$$y_1 = mx_1 + b$$
$$y_2 = mx_2 + b$$

From here, we can use substitution and elimination to find that $m = \frac{y_2-y_1}{x_2-x_1}$ and $b = y_1 - x_1 \cdot \frac{y_2-y_1}{x_2-x_1}$, so

$$y = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

which is the point-slope form of a line.

Note that the Euler-Lagrange equation only tells us that this is a stationary point of the arc length functional and not necessarily a minimum. To be more rigorous and show that the line is indeed the shortest path between the points, we could use the second variation of the functional. However, you could consider this result satisfactory just by intuition that any other curve increases length, and come to the conclusion that a line between the points is the geodesic.

Similar ideas can be used to solve the related problem of geodesics on a sphere. For example, spherical coordinates could be used instead of rectangular coordinates, and, after solving the Euler-Lagrange equation for the arc length functional of a curve on the sphere, you would find that the shortest path between two points on the sphere is an arc of the great circle through both points.

3.2. **The Brachistochrone Problem.**

Problem. What is the curve of fastest descent? That is, the curve between two points such that a particle sliding frictionlessly under the influence of gravity reaches the second point quickest.

Solution. Let $A = (0, h)$ and $B = (l, 0)$ be the two points in the plane, without loss of generality. There is also a constant gravitational force $\vec{g}$ acting in the downwards direction. To compare gravity with the particle's velocity v, we can use conservation of energy:

$$mgh = mgy + \frac{1}{2}mv^2$$

so $v = \sqrt{2g(h-y)}$.

Let the function $y(x)$ represent a possible curve between the two points. The quantity we would like to minimize is the time for a particle to travel along y, so we can set up a time functional:

$$J[y] = \int_A^B dt = \int_A^B \frac{ds}{v} = \int_{x_1}^{x_2} \frac{\sqrt{1 + (y')^2}}{\sqrt{2g(h-y)}}\, dx.$$

Since the Lagrangian $L(x, y, y') = \sqrt{\frac{1+(y')^2}{2g(h-y)}}$ does not involve x, we use the Beltrami identity instead of the Euler-Lagrange equation directly. So

$$\sqrt{\frac{1 + (y')^2}{2g(h-y)}} - y' \frac{y'}{\sqrt{(1 + (y')^2) \cdot 2g(h-y)}} = C,$$

and multiplying by $\sqrt{(1+(y')^2)\cdot 2g(h-y)}$ we have

$$\begin{aligned} 1+(y')^2-(y')^2 &= C\sqrt{(1+(y')^2)\cdot 2g(h-y)} \\ 1 &= C^2(1+(y')^2)\cdot 2g(h-y) \\ (1+(y')^2)(h-y) &= \frac{1}{C^2\cdot 2g} = C_1 \\ (h-y)+(y')^2(h-y) &= C_1 \\ y' &= \sqrt{\frac{C_1-(h-y)}{h-y}} \\ dx &= \sqrt{\frac{h-y}{C_1-(h-y)}}\ dy. \end{aligned}$$

We can now integrate both sides of the equation. The integration of the right-hand side is tedious, but the main idea is to use trigonometric substitution by letting $y = h - C_1 \sin^2(\theta/2)$. Eventually, we will find that

$$x = -\frac{C_1}{2}(\theta - \sin\theta) + K$$

for another constant K, and

$$y = h - \frac{C_1}{2}(1-\cos\theta)$$

which are parametric equations for the solution curve. These parametric equations happen to represent a cycloid curve, which means that the brachistochrone is a cycloid.

Now we can use the boundary conditions to find C_1 and K. We know that $y(0) = h$ and $y(l) = 0$, so $K = 0$ and

$$\begin{aligned} 2l &= -C_1(\theta_L - \sin\theta_L) \\ -2h &= -C_1(1-\cos\theta_L) \end{aligned}$$

where θ_L is the value of θ at point B. We could then solve for C_1 in terms of h and l, but that is also difficult and will not be done here for the sake of brevity.

A related problem is the Tautochrone/Isochrone Problem, which is a curve such that the time it takes for a point to reach the bottom under the influence of gravity is independent of its starting position. Interestingly, the solution to this problem is also a cycloid curve.

3.3. **Isoperimetric Problem.**

Problem. What is the shape of a constant-length curve which has the most area under it?

Solution. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be two points in the plane.

Let the function $y(x)$ represent a possible curve between the two points. Fix the length of y at ℓ, such that ℓ is greater than the distance between A and B. The quantity we would like to minimize is the area under y, so we can set up an area functional:

$$J[y] = \int_A^B dA = \int_{x_1}^{x_2} y\ dx.$$

However, note that we must also preserve the length functional:

$$I[y] = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx.$$

We have an additional functional applying a constraint, so we can define the Lagrangian as $L(x, y, y') = y + \lambda\sqrt{1 + (y')^2}$. Now we can apply the Euler-Lagrange equation

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 1 - \frac{d}{dx}\frac{\lambda y'}{\sqrt{1 + (y')^2}} = 0.$$

We can integrate with respect to x to get

$$x + C = \frac{y'}{\sqrt{1 + (y')^2}}$$

for some constant C, and further manipulate the equation to get that

$$y' = \frac{x + C}{\sqrt{\lambda^2 - (x + C)^2}}.$$

Now we can integrate this again using substitutions to find that $y + D = -\sqrt{\lambda^2 - (x + C)^2}$ for some constant D, which means

$$(x + C)^2 + (y + D)^2 = \lambda^2$$

which is the equation for a circle with radius λ and center $(-C, -D)$. Thus the curve which maximizes area is part of a circle.

In order to find the unknowns C, D, and λ, we can use the two boundary conditions along with the constraint equation on ℓ. To do this, we would basically just plug in values for the points A and B and to find C, D, and ℓ in terms of λ, then determine the nature of the circle from there.

A more general and slightly more complicated isoperimetric problem involves finding the planar shape that encloses the most area for a given perimeter, but the answer in this case is still a circle.

4. Conclusion

In summary, calculus of variations is a useful tool for finding minima and maxima of functionals. It encompasses important methods such as the Euler-Lagrange equation, along with other techniques such as the Beltrami Identity and Lagrange multipliers. Overall, it is very applicable to problems in other fields of mathematics, especially differential geometry.

Some other interesting problems in calculus of variations include the Catenary Problem (finding the curve that resembles the shape of a chain hanging between two points) and Plateau's Problem (finding minimal surfaces, which are surfaces of smallest area for a given boundary), among many others. Note that although the Euler-Lagrange is indeed a very powerful tool in calculus of variations, there are also other methods that can be used to solve problems in the field.

Apart from just mathematics, there are many applications of calculus of variations to other fields such as physics and engineering. For example, Fermat's Principle (that light follows path of shortest optical length), working with Dynamical Systems, and for Newtonian mechanics in general.

Acknowledgements

The author would like to thank Jacob Swenberg and Simon Rubinstein-Salzedo for feedback, support, and the opportunity to pursue this work as part of the Euler Circle Differential Geometry program.

References

[1] Jeff Calder. The calculus of variations. *University of Minnesota*, 40, 2020.
[2] Bernard Dacorogna. *Introduction to the Calculus of Variations*. World Scientific, 2024.
[3] James Ferguson. A brief survey of the history of the calculus of variations and its applications. *arXiv preprint math/0402357*, 2004.
[4] Irene Fonseca and Giovanni Leoni. Calculus of variations. 2018.
[5] Raju K George. The calculus of variations. 2014.
[6] Peter J Olver. The calculus of variations. 2021.

Euler Circle, Mountain View, CA 94040
Email address: `mail2mlrb2@gmail.com`