CALCULUS ON MANIFOLDS

RICHARD MAI

1. Introduction

Calculus on manifolds is the extension of familiar calculus concepts (derivatives, integrals, etc.) to more general spaces called *manifolds*. A manifold is essentially a space that looks locally like flat Euclidean space, allowing us to do calculus on "curved" spaces much as we do in $\mathbb{R}^n$. By developing calculus in this generalized setting, one can unify various results from multivariable calculus under a common framework. In particular, the celebrated *generalized Stokes' Theorem* encapsulates classical theorems such as the fundamental theorem of calculus, Green's theorem, Stokes' (surface) theorem, and the divergence theorem as special cases. In this exposition, we aim to give an intuitive overview of calculus on manifolds with minimal formalism, emphasizing geometric intuition and clear examples.

We will proceed as follows. First, we clarify what manifolds are and give some simple examples. Next, we discuss differentiation on manifolds, introducing the concept of tangent vectors and the derivative as a linear map. We then introduce *differential forms* as a natural language for integration on manifolds, leading to a statement of the generalized Stokes' Theorem. Throughout, we use the circle and sphere as running examples to illustrate the concepts. The discussion is intended for readers with only a basic background in multivariable calculus, and references for further accessible reading are provided at the end.

2. What is a Manifold?

Intuitively, a *manifold* is a space that, if you zoom in close enough, looks like flat Euclidean space. More formally, an n-dimensional manifold is a topological space in which every point has a neighborhood that can be continuously deformed (flattened out) into an open subset of $\mathbb{R}^n$. This means that although the space may be curved or complicated globally, locally it has coordinates and geometry like the familiar $\mathbb{R}^n$. For example, the surface of the Earth (a sphere) is a 2-dimensional manifold: each small patch of Earth's surface is nearly flat, and one can draw a local map (coordinate chart) that looks like a piece of the plane. Similarly, a curve like a circle is a 1-dimensional manifold (locally a line), and a torus (doughnut shape) is a 2-dimensional manifold that locally resembles a plane. [1]

Every manifold has an associated *dimension* n, which indicates how many coordinates are needed to parametrize a local neighborhood. To work with manifolds, mathematicians use collections of coordinate charts (an *atlas*) that cover the manifold. On overlapping regions of charts, coordinates transition by smooth transformations, ensuring the manifold has a well-defined smooth structure. In this paper we will restrict attention to *smooth manifolds*, where all the transition maps are differentiable, so that calculus makes sense. Intuitively, "smooth" means the manifold has no sharp corners or edges (contrast the smooth circle

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$y^2+x^2=1$ with a non-smooth corner at $y=|x|$) . With the basic idea of manifolds in place, we can now discuss how to differentiate and integrate on them. [2]

3. Differentiation and Tangent Vectors

Even on a curved manifold, we can talk about a function changing or a curve having a velocity. The key idea is the *tangent vector* at a point on the manifold. Geometrically, a tangent vector is like a direction in which one can move on the manifold starting from that point. If you imagine standing on the surface of a sphere, the directions you can walk form a plane tangent to the sphere at your feet. Formally, a tangent vector v at a point p can be defined as the velocity vector of some smooth curve $\gamma(t)$ on the manifold passing through p at $t=0$. This tangent vector lives in a linear space (the *tangent space* at p), which in an n-dimensional manifold is isomorphic to $\mathbb{R}^n$ (though it "sits" tangent to the manifold at p).

Now, given a smooth function f on a manifold M, how do we differentiate f at a point p? In multivariable calculus, one learns that the derivative (or differential) df at p is a linear map that takes a direction (vector) and returns the directional rate of change of f in that direction. The same idea holds on manifolds: $df(p)$ is a linear map on the tangent space at p. If v is a tangent vector at p, then $df(p)[v]$ (often denoted $v(f)$) gives the directional derivative of f along v. In local coordinates $(x^1,\ldots,x^n)$ around p, one can write v in components as $v=v^1\frac{\partial}{\partial x^1}\big|_p+\cdots+v^n\frac{\partial}{\partial x^n}\big|_p$. Correspondingly, $df(p)$ in coordinates is given by the gradient: $df(p)[v]=v^1\frac{\partial f}{\partial x^1}(p)+\cdots+v^n\frac{\partial f}{\partial x^n}(p)$. This generalizes the notion that in $\mathbb{R}^n$, the gradient ∇f dot a direction vector v gives the directional derivative. Viewing the derivative as a linear map on tangent vectors is a more fundamental perspective than thinking of it just as a gradient vector .

As a simple example, consider the unit circle S^1 (which is a 1-dimensional manifold) with a coordinate angle θ. A smooth function $f:S^1\to\mathbb{R}$ can be thought of as $f(\theta)$ in that coordinate. The tangent space at any point on S^1 is a line (tangent to the circle). A tangent vector can be identified with an angular direction $d/d\theta$. The derivative df at a point (with angle θ) applied to this tangent vector just yields $df(d/d\theta)=f'(\theta)$, the ordinary derivative with respect to the angle. This matches our intuition that the rate of change of f as we go around the circle is given by the derivative in the angle coordinate.

4. Differential Forms and Integration

In order to integrate on manifolds, we need a tool that generalizes the "integration element" (like dx, dx, dy, etc.) from calculus. The language of *differential forms* provides this in an elegant, coordinate-independent way. A differential form is an object that can be integrated over a manifold, and it inherently accounts for changes of coordinates (so that no Jacobian determinant needs to be manually inserted when changing variables).

Differential forms come in varying degrees. A 0-form is simply a function on the manifold (something that can be integrated in the 0-dimensional sense, i.e. evaluated at a point). A 1-form is an object that eats one tangent vector (at each point) and produces a number. For instance, in the plane $\mathbb{R}^2$, an example of a 1-form is $\omega=P(x,y),dx+Q(x,y),dy$, which when given a tangent direction (v_x,v_y) yields $P(x,y)v_x+Q(x,y)v_y$. A 2-form eats two tangent vectors and returns a number, and so on. These forms can be added, scaled, and "wedged" together (using the wedge product $\wedge$) to produce higher-degree forms. The wedge $\wedge$ is an antisymmetric product, meaning for example $dx\wedge dy=-,dy\wedge dx$, which is important for keeping track of orientation in integration.

Crucially, differential forms provide a systematic way to integrate on manifolds. A 1-form can be integrated along a curve (yielding a line integral), a 2-form can be integrated over a surface, etc. In fact, one can say that *differential forms exist to be integrated*: given a small k-dimensional piece of a manifold (spanned by k tangent vectors), a k-form will assign a number to that piece (like an oriented volume or area element), and integrating the form means summing up all those contributions over the manifold. Because forms transform under coordinate changes in a way that exactly cancels out the Jacobian factor, integrals of forms are independent of the choice of coordinates. This generalizes the change-of-variable formula from basic calculus.

For example, on the 2-dimensional sphere S^2, there is a natural 2-form Ω that represents area (in local latitude–longitude coordinates, Ω might be written as $R^2 \sin\phi, d\phi \wedge d\theta$ for a sphere of radius R). Integrating this 2-form Ω over the entire sphere $\int_{S^2} \Omega$ will yield the surface area $4\pi R^2$. As another example, on a curve (1-dimensional manifold), the ordinary line element ds is a 1-form; integrating ds along the curve gives its length. These examples show how differential forms generalize familiar integration concepts (length, area, volume) to arbitrary manifolds.

An important operation on differential forms is the *exterior derivative*, denoted d. The exterior derivative takes a k-form and produces a $(k+1)$-form, generalizing the notions of gradient, curl, and divergence from vector calculus. For instance, if f is a 0-form (function), then df is the 1-form representing its differential (whose action on a tangent vector gives the directional derivative of f). If $\omega = P, dx + Q, dy$ is a 1-form in the plane, then $d\omega$ is a 2-form given (in Cartesian coordinates) by $d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$. This $d\omega$ corresponds to the curl of the vector field (P, Q) in the plane. In general, $d(d\alpha) = 0$ for any form α (analogous to the fact that the curl of a gradient and the divergence of a curl are zero). With these tools in hand, we can now state the powerful general Stokes' theorem that ties differentiation and integration together.

5. The Generalized Stokes' Theorem

Stokes' theorem is the capstone that generalizes the fundamental theorem of calculus and all the major theorems of vector calculus to manifolds en.wikipedia.org . In its general form, it is a remarkably compact statement: if M is an oriented n-dimensional manifold with boundary ∂M, and ω is an $(n-1)$-form on M, then

$$\int_M d\omega \;=\; \int_{\partial M} \omega,$$

assuming the proper orientation on ∂M. In words, "the integral of the exterior derivative of ω over a manifold equals the integral of ω over the boundary of the manifold." This is often called the *Stokes–Cartan theorem.* It contains as special cases many familiar results en.wikipedia.org . For example:

- In one dimension, if $M = [a, b]$ is a line segment (a 1-manifold with boundary a, b) and $\omega = F(x)$ is a 0-form (function), then $d\omega = F'(x), dx$ and the formula reads $\int_a^b F'(x), dx = F(b) - F(a)$. This is the Fundamental Theorem of Calculus.
- In two dimensions, if M is a region in the plane with boundary curve ∂M, and $\omega = P, dx + Q, dy$ is a 1-form on M, then $d\omega = (\partial Q/\partial x - \partial P/\partial y), dx \wedge dy$. Stokes'

theorem in this case gives $\iint_M \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big), dx, dy = \oint_{\partial M} P, dx + Q, dy$, which is precisely *Green's Theorem* in the plane.

- In three dimensions, if M is a volume in $\mathbb{R}^3$ with closed surface boundary ∂M, and ω is a 2-form corresponding to a vector field $\mathbf{F}$ (through $\omega = F_x, dy \wedge dz + F_y, dz \wedge dx + F_z, dx \wedge dy$), then $d\omega$ is a 3-form corresponding to $(\nabla \cdot \mathbf{F}), dx \wedge dy \wedge dz$. Stokes' theorem becomes $\iiint_M (\nabla \cdot \mathbf{F}), dx, dy, dz = \iint_{\partial M} \mathbf{F} \cdot d\mathbf{S}$, which is the *Divergence Theorem* (Gauss's law). Similarly, the classical *Kelvin–Stokes theorem* (relating surface integrals of curl to line integrals around the boundary) is another instance.

Thus, the single formula $\int_M d\omega = \int_{\partial M} \omega$ simultaneously generalizes all these results. It underscores that differentiation and integration are two sides of the same coin: the integration of a "derivative" over a region is determined by the values on the boundary. This unifying perspective is one of the biggest payoffs of calculus on manifolds. [3]

6. Simple Examples: Circle and Sphere

We now revisit our two basic examples, the circle and the sphere, to see calculus on manifolds in action.

The Circle S^1. The unit circle S^1 can be viewed as a 1-dimensional manifold (without boundary). It can be covered by two coordinate charts (e.g. an angle coordinate θ on $[0, 2\pi)$ with a cut, or two overlapping parametrizations from 0 to 2π). On S^1, consider a smooth function $f(\theta)$ (as earlier). The derivative df is a 1-form on the circle; in fact, in the θ coordinate it is $f'(\theta), d\theta$. If we integrate the derivative around the entire circle, we get $\int_{S^1} f'(\theta), d\theta$. By the 1-dimensional Stokes' theorem (essentially the fundamental theorem of calculus on a closed loop), this should equal $f(\theta)\big|\theta = 2\pi - f(\theta)\big|\theta = 0$. But since $\theta = 0$ and $\theta = 2\pi$ correspond to the same point on the circle, the result is zero. This illustrates that $\oint_{S^1} df = 0$ for any exact 1-form df, consistent with the general fact that a closed manifold (one without boundary, like S^1) has $\partial M = \emptyset$, so $\int_M d\omega = \int_\emptyset \omega = 0$.

As a more concrete example, consider the 1-form $\omega = d\theta$ on the circle (which is the derivative of the identity angle function). Integrating this around S^1 gives $\int_{S^1} d\theta = 2\pi$. In this case ω is not globally the derivative of a single-valued function on S^1 (since θ is multi-valued or discontinuous), so Stokes' theorem does not directly apply to ω globally. However, we can cover the circle with two charts and observe that the discrepancy (the jump in θ) at the cut yields the 2π. This ties into more advanced topological ideas (the form $d\theta$ represents a nontrivial cohomology class on the circle), but the main takeaway for our purposes is understanding how integration on a simple closed manifold works.

The Sphere S^2. The 2-dimensional sphere S^2 (unit sphere in $\mathbb{R}^3$) is a 2-manifold without boundary. We already mentioned that one can integrate a suitable 2-form on S^2 to get the surface area. For instance, in spherical coordinates, an area 2-form is $\Omega = \sin\phi, d\phi \wedge d\theta$ on S^2 of radius 1. Integrating, $\iint_{S^2} \sin\phi, d\phi, d\theta = 4\pi$, the well-known surface area. Now, since S^2 has no boundary, Stokes' theorem tells us that for any 1-form ω defined on all of S^2, $\iint_{S^2} d\omega = \int_{\partial S^2} \omega = \int_\emptyset \omega = 0$. For example, take a vector field on the sphere and the associated flux 2-form ω (as in the divergence theorem analogy). The total "outflow" through the closed surface S^2 must be zero if the field has no sources inside. This is consistent with the fact that, say, the divergence of a curl is zero (any field that is globally the curl of another has zero net flux). In a more geometric vein, consider a 1-form α on S^2 that is like a "potential" for some 2-form $\omega = d\alpha$ (not every 2-form on S^2 has a global potential 1-form, but if it does, we call ω an exact form). Stokes' theorem then guarantees the integral of ω over S^2 is zero.

These simple examples illustrate the general principles on familiar objects. The circle and sphere also hint at deeper topological phenomena (like the inability to define a global angle coordinate on S^1 without a cut, or the existence of divergence-free vector fields on S^2 that are not curls globally). However, for our purposes, they serve to build intuition: manifolds allow calculus to be done *in situ* on curved spaces, and Stokes' theorem will faithfully reproduce and generalize the results we expect.

7. Concluding Thoughts

We have given a whirlwind tour of calculus on manifolds, focusing on the big picture and geometric intuition. The concept of a manifold allows us to extend calculus beyond flat space, and differential forms provide a powerful and elegant language for integration that makes formulas coordinate-free. The generalized Stokes' theorem stands out as a unifying theorem that includes as special cases nearly every major theorem of calculus, illustrating the unity underlying differentiation and integration in all dimensions.

There is much more to explore beyond this brief introduction. For instance, the ideas here lead into de Rham cohomology, which connects differential forms to the topological features of manifolds (as briefly hinted by the examples on S^1 and S^2). But even within classical calculus on manifolds, mastering these concepts can greatly deepen one's understanding of multivariable calculus and differential geometry. We hope this exposition has provided an accessible starting point. For further reading, the following resources are recommended.

References

[1] D. Bachman. *A Geometric Approach to Differential Forms*. Birkhäuser, 2 edition, 2012.
[2] J. P. Fortney. *A Visual Introduction to Differential Forms and Calculus on Manifolds*. Birkhäuser, 2018.
[3] M. Spivak. *Calculus on Manifolds*. Benjamin, 1965.