Euler Circle

Expository Paper on Symplectic Geometry

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1) **Introduction:** In simple terms, symplectic geometry is a method of measuring oriented area, rather than lengths and angles as seen in Riemannian geometry. Originating from classical mechanics, where phase space is described by positions and momenta, symplectic geometry helps define the geometric framework of Hamiltonian mechanics. Moreover, symplectic geometry is essential in geometric quantization, the process connecting quantum mechanics to classical mechanics. Symplectic geometry's uses span far beyond physics. It is relevant in other fields such as computer science and engineering, being a powerful tool in studying algebraic geometry, representation theory, and dynamic systems.

2) **Basics:** Symplectic geometry, also sometimes referred to as symplectic topology, is a branch of differential geometry that studies symplectic manifolds—differentiable smooth manifolds that have a closed, nondegenerate differentiable 2-form, known as the symplectic structure, whilst the 2-form itself is referred to as the symplectic form. Firstly, differentiable k-forms on a manifold are a section of the k^{th} exterior power of the cotangent bundle, where the cotangent bundle at a point consists of all linear maps from the tangent space at that point to $\mathbb{R}$. Here are a few examples:

    0-forms: Smooth functions on the manifold

1-forms: A function that assigns a covector to each point on the manifold
2-forms: A type of function that is applied on two factors and measures infinitesmal oriented area—infinitely small area that has direction/orientation.

For a differentiable k-form to be a closed form, it must have its exterior derivative be 0. Additionally, the exterior derivative of the k-form is a $k+1$-form. Typically the symplectic form is referred to as ω and the smooth manifold is referred to as M. The pair (M, ω) is called a symplectic manifold. ω's nondegeneracy implies that M must be even-dimensional and that the tangent and cotangent bundles are isomorphic.

Remark: The condition that ω must be closed ensures that potential forms exist locally. Additionally, it eventually leads to Darboux's theorem, one of the most notable theorems in symplectic geometry.

3) **Symplectic geometry vs. Riemannian geometry:** Symplectic geometry and Riemannian geometry have many similarities and differences. Firstly, Riemannian geometry is the study of differentiable manifolds that have non-degenerate, symmetric 2-tensors. 2-tensors are a type of metric tensor—a structure on a manifold that defines distance and angles as the dot product of Euclidean space. One example of Riemannian and symplectic geometry's differences is that not every differentiable manifold can take a symplectic form due to the topological restrictions, such as that all symplectic manifolds must be of even dimension. On the other hand, one example of a similarity

between the two fields is between Riemannian geometry's geodesics and symplectic geometry's pseudoholomorphic curves. In Riemannian geometry, geodesics are locally curves of shortest length while in symplectic geometry, pseudoholomorphic curves are surfaces of minimal area.

4) **Darboux's Theorem:** Darboux's theorem is one of the most fundamental theorems in symplectic geometry, as are Weinstein's Neighborhood Theorem and Gromov's Nonsqueezing Theorem, but in this paper only Darboux's theorem will be focused on.

Theorem (Darboux's Theorem): Let (M, ω) be a symplectic manifold and let $p \in M$. There exist coordinates locally $(q_1, q_2, q_3, \dots, q_n, p_1, p_2, p_3, \dots, p_n)$ around p such that

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$$

in these coordinates.

In simple terms, Darboux's theorem expresses the simple but core idea that locally, all symplectic manifolds look the same. Firstly, it is necessary to understand the wedge ($\wedge$) product and canonical coordinates.

Definition: The wedge product is an antisymmetric bilinear operation. The wedge product is defined on differential forms and essentially takes a k-form α and a separate l-form β which it produces a $(k+l)$-form, namely $\alpha \wedge \beta$. The antisymmetry of the wedge product means that $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$.

Definition: Generally, canonical coordinates are a system of coordinates that brings the coordinates of a geometric setting to a standard, simpler form. In symplectic geometry, the coordinates $(q_1, q_2, q_3, \ldots, q_n, p_1, p_2, p_3, \ldots, p_n)$ in Darboux's theorem are canonical coordinates.

As stated before, Darboux's theorem is one of the most important theorems in symplectic geometry. Firstly, Darboux's theorem ensures that there are no local invariants—numerical, algebraic, or topological quantities that remain unchanged when put through a symplectic structure—unlike how in Riemannian geometry there are necessary local invariants such as curvature, yet another difference between symplectic geometry and Riemannian geometry. The fact that there are no local invariants due to Darboux's theorem leads to the uniformity of symplectic manifolds locally. This uniformity allows symplectic structures to be understood by understanding just the standard form. This is especially useful in Hamiltonian mechanics, as it enables a standard, local coordinate system to be used in phase space, essentially a 2n-dimensional space.

References

[1] Yang, Jay-Hyun. "Remarks on Symplectic Geometry." arXiv. Last modified October 11, 2019. arXiv. https://arxiv.org/abs/1910.05011.

[2] Institute, Fields. Symplectic Geometry Class 1. Youtube, 2021. Film. https://youtu.be/Eh2x2BthLp4

[3] Differential Geometry 2023 - Lecture 17 . Youtube, 2023. https://youtu.be/nPO-C2dQoUY

[4] Calaque, Damien. "Derived symplectic geometry." arXiv. Last modified February 24, 2024. arXiv. https://arxiv.org/abs/2308.04210.

[5] De Silva, Anna C. "Lectures on Symplectic Geometry." Department of Mathematics | ETH Zurich. Last modified January , 2006. https://people.math.ethz.ch/.

[6] Wikipedia. Last modified May 25, 2025. https://en.wikipedia.org/wiki/Darboux%27s_theorem#.