De Rham Cohomology

Prashanth Prabhala
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1 Introduction

De Rham cohomology connects differential forms on smooth manifolds to topological invariants. The basic idea is that when you differentiate a differential form twice, you always get zero. This property $d^2 = 0$ lets us build cohomology groups that detect holes and other topological features of a manifold.

Here's a simple example. On the circle S^1, the 1-form $d\theta$ (where θ is the angular coordinate) is closed since $d(d\theta) = 0$. But there's no function f on S^1 with $df = d\theta$. Why? Because θ isn't well-defined as a function on the whole circle. It jumps by 2π when you go around. This failure of $d\theta$ to be exact reflects the fact that S^1 has a hole in it.

The de Rham cohomology groups organize these observations. They measure which closed forms fail to be exact, and this failure corresponds to topological properties of the manifold. The de Rham theorem makes this precise: these cohomology groups are isomorphic to the singular cohomology groups with real coefficients.

2 Differential Forms and Pullbacks

Let M be a smooth n-dimensional manifold. A differential k-form assigns to each point $p \in M$ an alternating k-linear map on the tangent space T_pM, and this assignment varies smoothly with p.

Definition 1. The space of smooth k-forms on M is denoted $\Omega^k(M)$. For $k = 0$, we have $\Omega^0(M) = C^\infty(M)$, the smooth functions on M.

In local coordinates $(x^1, \ldots, x^n)$, any k-form can be written as

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where the coefficients $\omega_{i_1 \cdots i_k}$ are smooth functions.

The wedge product combines a k-form and an ℓ-form to give a $(k+\ell)$-form. It satisfies

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$$

for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$.

Definition 2 (Exterior derivative)**.** The exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is the unique linear operator satisfying:

1. On functions $f \in \Omega^0(M)$, we have $df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$ in local coordinates.

2. $d(d\omega) = 0$ for any form ω.

3. $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ for $\alpha \in \Omega^k(M)$.

To compute d in coordinates, if $\omega = \sum_I \omega_I \, dx^I$ using multi-index notation, then

$$d\omega = \sum_I \sum_{j=1}^{n} \frac{\partial \omega_I}{\partial x^j} \, dx^j \wedge dx^I.$$

The key property is $d^2 = 0$. You can check this directly using the equality of mixed partials and the antisymmetry of the wedge product.

Definition 3 (Pullback)**.** If $f : M \to N$ is smooth and $\omega \in \Omega^k(N)$, the pullback $f^*\omega \in \Omega^k(M)$ is defined by

$$(f^*\omega)_p(v_1, \dots, v_k) = \omega_{f(p)}(df_p(v_1), \dots, df_p(v_k)).$$

This operation is linear, commutes with wedge products and with d, and is fundamental to the functoriality of de Rham cohomology.

3 De Rham Cohomology

The exterior derivative gives us a sequence

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0.$$

Since $d^2 = 0$, the image of each d is contained in the kernel of the next one.

Definition 4. A k-form ω is called:

- closed if $d\omega = 0$
- exact if $\omega = d\eta$ for some $(k-1)$-form η

Every exact form is closed, but not every closed form is exact. The obstruction to a closed form being exact is what cohomology measures.

Definition 5. The k-th de Rham cohomology group is

$$H^k_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^k \to \Omega^{k+1})}{\operatorname{im}(d : \Omega^{k-1} \to \Omega^k)}.$$

4 Integration of Forms

To use integration in cohomology, we need orientation.

Definition 6 (Orientation)**.** A smooth n-manifold M is orientable if it admits a nowhere-vanishing top-degree form. Choosing such a form fixes an orientation.

Definition 7 (Integration of top forms)**.** If M is an oriented n-manifold and $\omega \in \Omega^n_c(M)$ has compact support, then

$$\int_M \omega$$

is defined by patching local integrals through partitions of unity. For embedded oriented submanifolds $\sigma : \Delta^k \to M$, one defines

$$\int_\sigma \omega = \int_{\Delta^k} \sigma^*\omega.$$

Theorem 1 (Stokes)**.** *If M is oriented with boundary and $\omega \in \Omega_c^{n-1}(M)$, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

This theorem underlies the cochain map construction in the de Rham theorem.

5 Basic Examples

Example 1 (Euclidean space)**.** For $M = \mathbb{R}^n$, we have

$$H^k_{\mathrm{dR}}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{if } k \geq 1. \end{cases}$$

Theorem 2 (Poincaré Lemma)**.** *If $U \subseteq \mathbb{R}^n$ is star-shaped, every closed k-form on U with $k \geq 1$ is exact.*

Example 2 (The circle)**.** For S^1,

$$H^k_{\mathrm{dR}}(S^1) = \begin{cases} \mathbb{R} & k = 0, 1 \\ 0 & k \geq 2. \end{cases}$$

Example 3 (The n-sphere)**.** For S^n with $n \geq 1$,

$$H^k_{\mathrm{dR}}(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

6 The Poincaré Lemma in Detail

Let's prove the Poincaré lemma for a star-shaped domain $U \subseteq \mathbb{R}^n$ with center at the origin. The idea is to construct a homotopy operator $K : \Omega^k(U) \to \Omega^{k-1}(U)$ such that

$$dK + Kd = \mathrm{id} - \pi^*$$

where $\pi : U \to \{0\}$ is the constant map. Since π^* is zero on forms of positive degree, this gives $dK\omega = \omega$ for any closed k-form ω with $k \geq 1$.

For a k-form $\omega = \sum_I \omega_I(x)\, dx^I$, define

$$(K\omega)(x) = \sum_{|I|=k} \left(\int_0^1 t^{k-1} \sum_{j=1}^n x^j \omega_{jI}(tx)\, dt \right) dx^I$$

where ω_{jI} means the coefficient of $dx^j \wedge dx^I$ in ω.

We can verify by direct calculation that $dK + Kd$ gives the identity on forms of positive degree. The key is that the star-shaped property ensures the integral is well-defined.

7 Mayer–Vietoris Sequence

The Mayer–Vietoris exact sequence is central for computing de Rham cohomology. If $M = U \cup V$ with U, V open, then there is a long exact sequence

$$\cdots \to H^{k-1}_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta} H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \to H^k_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta} \cdots$$

This allows computation of cohomology by covering M with simple pieces.

8 The de Rham Theorem

Let $C_k(M)$ denote the group of singular k-chains and ∂ the boundary operator. If $\omega \in \Omega^k(M)$ and $\sigma : \Delta^k \to M$ is a singular k-simplex, define

$$I(\omega)(\sigma) \int_\sigma \omega = \int_{\Delta^k} \sigma^* \omega.$$

This gives a cochain $I(\omega) \in C^k(M;\mathbb{R}) = \mathrm{Hom}(C_k(M), \mathbb{R})$.

Proposition 1 (Stokes implies cochain map)**.** *For all $\omega \in \Omega^{k-1}(M)$,*

$$\delta(I(\omega)) = I(d\omega),$$

where δ is the singular coboundary. Equivalently, $I : \Omega^\bullet(M) \to C^\bullet(M;\mathbb{R})$ is a cochain map.

Proof sketch. By Stokes' theorem, $\int_{\partial\sigma} \omega = \int_\sigma d\omega$ for each singular simplex σ. Unwinding definitions gives $\delta(I(\omega)) = I(d\omega)$. □

Therefore I descends to cohomology:

$$\mathcal{I} \colon H^k_{\mathrm{dR}}(M) \longrightarrow H^k(M;\mathbb{R}).$$

Theorem 3 (de Rham)**.** *For any smooth manifold M, the map $\mathcal{I}$ is an isomorphism for all k. It is natural with respect to smooth maps and is an isomorphism of graded commutative $\mathbb{R}$-algebras:*

$$\mathcal{I}([\alpha] \wedge [\beta]) = \mathcal{I}([\alpha]) \smile \mathcal{I}([\beta]).$$

Idea of proof. Choose a good cover of M by contractible opens. On each open set, Poincaré lemma identifies de Rham and singular cohomology (both vanish in positive degrees). Using Mayer–Vietoris/Čech cohomology and partitions of unity, one promotes these local identifications to a global one. Alternatively, Whitney forms give an explicit inverse to $\mathcal{I}$. □

Remark 1 (Naturality). If $f \colon M \to N$ is smooth, then for all $[\omega] \in H^\bullet_{\mathrm{dR}}(N)$,

$$\mathcal{I}\big(f^*[\omega]\big) = f^*\big(\mathcal{I}([\omega])\big) \in H^\bullet(M;\mathbb{R}).$$

9 Applications

9.1 Cohomology ring of the torus

Let $T^n = (S^1)^n$ with angular coordinates $(\theta_1, \ldots, \theta_n)$. The 1-forms $d\theta_i$ are closed and not exact; their classes $x_i = [d\theta_i] \in H^1_{\mathrm{dR}}(T^n)$ are linearly independent. Wedge products give

$$x_{i_1} \wedge \cdots \wedge x_{i_k} = [\, d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_k}\,],$$

and these span $H^k_{\mathrm{dR}}(T^n)$. Hence

$$H^*_{\mathrm{dR}}(T^n) \cong \Lambda^*\big(\mathbb{R}^n\big),$$

the exterior algebra on n degree-1 generators. Under Theorem 3, this matches the singular cohomology ring with cup product.

9.2 Degree of a map

If M, N are closed, connected, oriented n-manifolds and $f\colon M \to N$ is smooth, the *degree* $\deg(f) \in \mathbb{Z}$ is characterized by

$$\int_M f^{\omega=\deg(f)\int_N \omega} \quad \text{for every top-degree form } \omega \text{ on } N.$$

Equivalently, $f^{:H^n_{\mathrm{dR}}(N)\to H^n_{\mathrm{dR}}(M)}$ is multiplication by $\deg(f)$ on the orientation classes. For $f\colon S^1 \to S^1$, $f(e^{i\theta}) = e^{ik\theta}$ has $\deg(f) = k$ since

$$\int_{S^1} f^{\left(\frac{d\theta}{2\pi}\right)=\frac{1}{2\pi}\int_0^{2\pi} k\,d\theta=k.}$$

9.3 Path-independence and exactness

Let ω be a closed 1-form on a connected manifold M.

- If $\int_\gamma \omega = 0$ for every loop γ in M, then $[\omega] = 0$ in $H^1_{\mathrm{dR}}(M)$ and $\omega = df$ for some f (global potential). Thus line integrals of ω are path-independent.
- In particular, if M is simply connected, then $H^1_{\mathrm{dR}}(M) = 0$ and every closed 1-form is exact.

9.4 Orientability and top cohomology

If M is a closed n-manifold, then

$$H^n_{\mathrm{dR}}(M) \cong \begin{cases} \mathbb{R} & \text{if } M \text{ is oriented and connected,} \\ 0 & \text{if } M \text{ is nonorientable.} \end{cases}$$

When M is oriented, any volume form represents the orientation class. Integration defines a nondegenerate pairing

$$H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) \longrightarrow \mathbb{R}, \qquad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta,$$

which is the de Rham incarnation of Poincaré duality.

Example 4 (Recovering known computations)**.** For T^2, $H^0 = \mathbb{R}$, $H^1 = \mathbb{R}\langle [d\theta_1], [d\theta_2] \rangle$, $H^2 = \mathbb{R}\langle [d\theta_1 \wedge d\theta_2] \rangle$, and wedge corresponds to cup under Theorem 3. For S^n, the top class is the (normalized) volume form.

Remark 2 (Mayer–Vietoris in practice)*.* Combined with the Poincaré lemma, the Mayer–Vietoris sequence lets you compute $H^\bullet_{\mathrm{dR}}$ by cutting M into contractible pieces and tracking overlaps. The result automatically agrees with singular cohomology by Theorem 3.

10 Conclusion

De Rham cohomology provides a bridge between the analytic world of differential forms and the topological world of invariants. Starting from $d^2 = 0$, we built cohomology groups that record when closed forms fail to be exact. Through spheres, circles, and tori, we saw how these groups detect the presence of holes and higher-dimensional features.

The de Rham theorem makes this precise: differential forms capture the same information as singular cohomology with real coefficients. Applications such as computing the cohomology ring of the torus, defining the degree of a map, and understanding orientability illustrate how geometry, topology, and analysis connect in this theory.

It also links to Hodge theory, characteristic classes, and modern approaches in algebraic topology and algebraic geometry.

References

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