An Exposition on Minimal Surfaces

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1 Introduction to Minimal Surfaces

Minimal surfaces are among the most fascinating objects in differential geometry, representing a deep connection between pure geometry and physical phenomena. They arise naturally in the study of soap films, which, due to surface tension, configure themselves to minimize their surface area for a given boundary. This leads to the famous **Plateau's Problem**: to find the surface of least area spanning a given boundary curve. The solutions to this problem are minimal surfaces. However, a common misconception about Plateau's Problem is that all minimal surfaces must be a solution for some boundary curve. Instead, minimal surfaces are only locally minimal, as we will show with the variational definition of minimal surfaces.

Formally, we approach the study of surfaces through their curvature. For a regular **surface patch** $\sigma : U \to \mathbb{R}^3$, where U is an open set in $\mathbb{R}^2$, we can define two **principal curvatures**, κ_1 and κ_2, at each point in its image. These represent the maximum and minimum normal curvatures of curves passing through the point on the surface.

Definition 1.1 (Mean and Gaussian Curvature)**.** *For a surface patch with principal curvatures κ_1 and κ_2, the* ***mean curvature*** *(H) and* ***Gaussian curvature*** *(K) are defined as:*

$$H = \frac{\kappa_1 + \kappa_2}{2}$$
$$K = \kappa_1 \kappa_2$$

This leads to the primary geometric definition of a minimal surface.

Definition 1.2 (Minimal Surface Patch - Curvature Definition)**.** *A regular surface patch $\sigma : U \to \mathbb{R}^3$ is a* ***minimal surface patch*** *if its mean curvature is identically zero, i.e., $H = 0$ at every point. A surface is called minimal if it can be covered by an atlas of minimal surface patches.*

An immediate consequence is that for a minimal surface patch, $\kappa_1 = -\kappa_2$. This implies that at any non-flat point, the surface is locally shaped like a saddle.

The second, equivalent definition comes from the calculus of variations and relates to the physical property of minimizing area.

Definition 1.3 (Minimal Surface Patch - Variational Definition)**.** *A regular surface patch $\sigma : U \to \mathbb{R}^3$ is a* ***minimal surface patch*** *if it is a critical point for the area functional.*

This means that any small, compactly supported normal variation of the patch results in a positive change in its surface area.

2 Equivalence of Definitions

We now prove that the geometric definition ($H = 0$) is equivalent to the variational definition (being a critical point for the area functional).

Theorem 2.1. *A regular surface patch $\sigma : U \to \mathbb{R}^3$ is a critical point for the area functional under compactly supported normal variations if and only if its mean curvature H is identically zero.*

Proof. Let $\sigma(u, v)$ be a regular surface patch. Let $\mathbf{N}$ be the unit normal vector given (u, v), and epsilon be some small real number used to show the difference between σ and σ_ϵ. We consider a normal variation of the surface given by:

$$\sigma_\epsilon(u, v) = \sigma(u, v) + \epsilon\phi(u, v)\mathbf{N}(u, v)$$

where $\phi(u, v)$ is a smooth function with compact support within the domain of the patch. The area of the varied surface is given by the functional:

$$A(\epsilon) = \iint \sqrt{E_\epsilon G_\epsilon - F_\epsilon^2}\, du\, dv$$

where $E_\epsilon, F_\epsilon, G_\epsilon$ are the coefficients of the first fundamental form of σ_ϵ.

The surface patch is a critical point for the area if the first variation of the area is zero, i.e., $\frac{dA}{d\epsilon}\big|_{\epsilon=0} = 0$.

We compute the derivatives of the tangent vectors of σ_ϵ:

$$\frac{\partial \sigma_\epsilon}{\partial u} = \sigma_u + \epsilon(\phi_u \mathbf{N} + \phi \mathbf{N}_u)$$
$$\frac{\partial \sigma_\epsilon}{\partial v} = \sigma_v + \epsilon(\phi_v \mathbf{N} + \phi \mathbf{N}_v)$$

The Weingarten map, $d\mathbf{N}$, relates the derivative of the normal to the tangent vectors: $\mathbf{N}_u = d\mathbf{N}(\sigma_u)$ and $\mathbf{N}_v = d\mathbf{N}(\sigma_v)$.

Now we compute the derivatives of the coefficients of the first fundamental form at $\epsilon = 0$:

$$\left.\frac{dE}{d\epsilon}\right|_{\epsilon=0} = 2\langle \sigma_u, \phi_u \mathbf{N} + \phi \mathbf{N}_u \rangle = 2\phi\langle \sigma_u, \mathbf{N}_u \rangle = -2\phi L$$

where $L = -\langle \sigma_u, \mathbf{N}_u \rangle = \langle \sigma_{uu}, \mathbf{N} \rangle$ is a coefficient of the second fundamental form. Similarly,

$$\left.\frac{dG}{d\epsilon}\right|_{\epsilon=0} = 2\phi\langle \sigma_v, \mathbf{N}_v \rangle = -2\phi N$$
$$\left.\frac{dF}{d\epsilon}\right|_{\epsilon=0} = \phi(\langle \sigma_u, \mathbf{N}_v \rangle + \langle \sigma_v, \mathbf{N}_u \rangle) = -2\phi M$$

The area element is $dA = \sqrt{EG - F^2}\, du\, dv$. The derivative of the area element integrand is:

$$\begin{aligned} \frac{d}{d\epsilon}\sqrt{EG - F^2}\bigg|_{\epsilon=0} &= \frac{1}{2\sqrt{EG - F^2}}\left(\frac{dE}{d\epsilon}G + E\frac{dG}{d\epsilon} - 2F\frac{dF}{d\epsilon}\right)\bigg|_{\epsilon=0} \\ &= \frac{1}{2\sqrt{EG - F^2}}(-2\phi LG - 2\phi EN + 4\phi FM) \\ &= -\phi\frac{LG + EN - 2FM}{\sqrt{EG - F^2}} \end{aligned}$$

Recall that the mean curvature is $H = \frac{EN+GL-2FM}{2(EG-F^2)}$. Thus,

$$\frac{d}{d\epsilon}\sqrt{EG - F^2}\bigg|_{\epsilon=0} = -2H\phi\sqrt{EG - F^2}$$

The first variation of area is then:

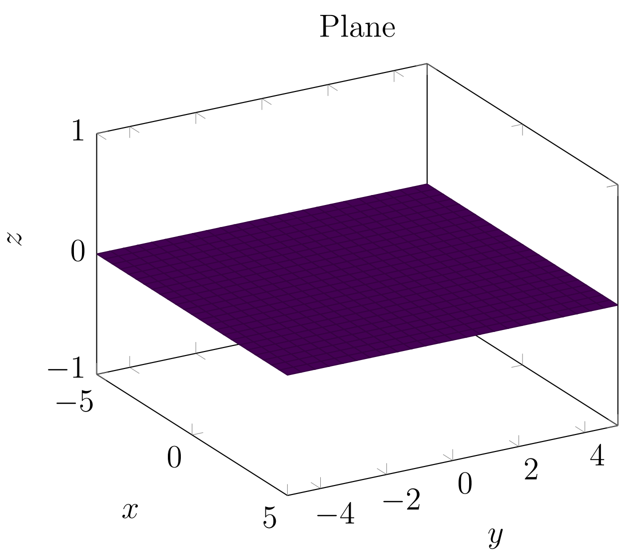
$$\frac{dA}{d\epsilon}\bigg|_{\epsilon=0} = \iint -2H\phi\sqrt{EG - F^2}\, du\, dv = \iint -2H\phi\, dA$$

For this to be zero for all compactly supported smooth functions ϕ, the fundamental lemma of calculus of variations implies that the integrand must be identically zero. Therefore, $-2H = 0$, which means $H = 0$. □

3 A Gallery of Minimal Surfaces

We now demonstrate that several classical surface patches are minimal by showing they satisfy the condition $H = 0$.

3.1 The Plane



The plane is the most trivial example. We consider a patch on a plane given by $z = c$.

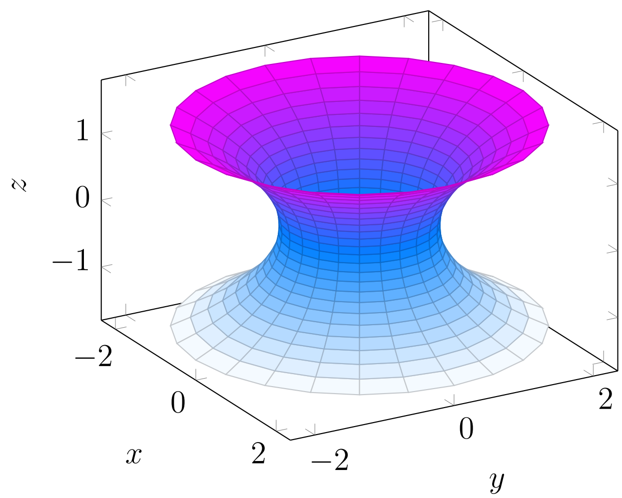
- **Parametrization:** $\sigma(u, v) = (u, v, c)$.
- **Derivatives:** $\sigma_u = (1, 0, 0)$ and $\sigma_v = (0, 1, 0)$. All second partial derivatives are zero: $\sigma_{uu} = \sigma_{uv} = \sigma_{vv} = \mathbf{0}$.
- **Normal Vector:** $\mathbf{N} = \sigma_u \times \sigma_v = (0, 0, 1)$.
- **Fundamental Forms:** The coefficients of the first fundamental form are $E = 1, F = 0, G = 1$. The coefficients of the second fundamental form are $L = \langle \sigma_{uu}, \mathbf{N} \rangle = 0$, $M = \langle \sigma_{uv}, \mathbf{N} \rangle = 0$, and $N = \langle \sigma_{vv}, \mathbf{N} \rangle = 0$.
- **Mean Curvature:**

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} = \frac{1(0) + 1(0) - 2(0)(0)}{2(1 - 0)} = 0$$

Thus, this patch of the plane is minimal.

3.2 The Catenoid

Catenoid



The catenoid is the surface of revolution generated by a catenary curve. A patch on this surface is given by:

- **Parametrization:** Let the profile curve be $x = a\cosh(z/a)$. Rotating this around the z-axis gives the patch:

$$\sigma(u, v) = (a\cosh(v/a)\cos u, a\cosh(v/a)\sin u, v)$$

- **Fundamental Forms:** For a surface of revolution patch $\sigma(u,v) = (f(v)\cos u, f(v)\sin u, v)$ where $f(v) = a\cosh(v/a)$, the coefficients are:

$$\begin{aligned} E &= f(v)^2 = a^2\cosh^2(v/a) \\ F &= 0 \\ G &= 1 + f'(v)^2 = 1 + \sinh^2(v/a) = \cosh^2(v/a) \\ L &= -f(v) = -a\cosh(v/a) \\ M &= 0 \\ N &= \frac{f''(v)}{\sqrt{1+f'(v)^2}} = \frac{(1/a)\cosh(v/a)}{\cosh(v/a)} = 1/a \end{aligned}$$

- **Mean Curvature:**

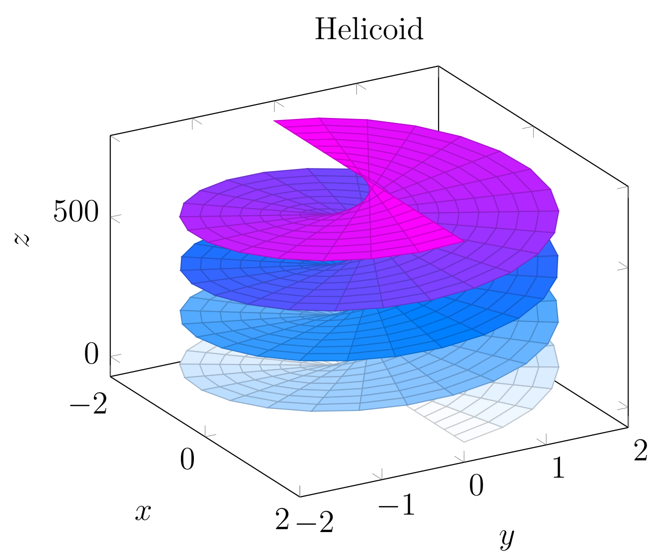
$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} = \frac{(a^2\cosh^2(v/a))(1/a) + (\cosh^2(v/a))(-a\cosh(v/a))}{2(a^2\cosh^4(v/a))}$$

  This simplifies to:

$$H = \frac{a\cosh^2(v/a) - a\cosh^2(v/a)}{2(a^2\cosh^4(v/a))} = 0$$

The catenoid patch is minimal.

3.3 The Helicoid



The helicoid is a ruled surface that resembles a screw, defined by the following patch:

- **Parametrization:** For a constant c,

$$\sigma(u,v) = (v\cos u, v\sin u, cu)$$

- **Derivatives:**

$$\begin{aligned}\sigma_u &= (-v\sin u, v\cos u, c)\\ \sigma_v &= (\cos u, \sin u, 0)\\ \sigma_{uu} &= (-v\cos u, -v\sin u, 0)\\ \sigma_{uv} &= (-\sin u, \cos u, 0)\\ \sigma_{vv} &= (0,0,0)\end{aligned}$$

- **Fundamental Forms:**

$$\begin{aligned}E &= \langle \sigma_u, \sigma_u\rangle = v^2 + c^2\\ F &= \langle \sigma_u, \sigma_v\rangle = 0\\ G &= \langle \sigma_v, \sigma_v\rangle = 1\end{aligned}$$

The unit normal vector is $\mathbf{N} = \frac{\sigma_u\times\sigma_v}{\|\sigma_u\times\sigma_v\|} = \frac{1}{\sqrt{v^2+c^2}}(-c\sin u, c\cos u, -v)$.

$$\begin{aligned}L &= \langle \sigma_{uu}, \mathbf{N}\rangle = 0\\ M &= \langle \sigma_{uv}, \mathbf{N}\rangle = \frac{-c}{\sqrt{v^2+c^2}}\\ N &= \langle \sigma_{vv}, \mathbf{N}\rangle = 0\end{aligned}$$

- **Mean Curvature:**

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} = \frac{(v^2+c^2)(0) + 1(0) - 2(0)M}{2(v^2+c^2)} = 0$$

The helicoid patch is minimal.

3.4 Scherk's Second Surface

Scherk's surface is a singly periodic minimal surface, typically given by an implicit equation.

- **Implicit Equation:** $\sin(z) - \sinh(x)\sinh(y) = 0$.
- **Proof of Minimality:** We can treat a patch of the surface as a graph $z = f(x, y)$ and show it satisfies the minimal surface equation:

$$(1 + f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1 + f_x^2)f_{yy} = 0$$

By implicit differentiation of the surface equation, we find the necessary partial derivatives:

$$z_x = \frac{\cosh(x)\sinh(y)}{\cos(z)} \qquad z_y = \frac{\sinh(x)\cosh(y)}{\cos(z)}$$

$$z_{xx} = \tan(z)(1 + z_x^2) \qquad z_{yy} = \tan(z)(1 + z_y^2)$$

Substituting the expressions for z_{xx} and z_{yy} into the minimal surface equation leads to a large expression, which can be simplified with identities $\sin(z) = \sinh(x)\sinh(y)$ and $\cosh^2(\theta) = 1 + \sinh^2(\theta)$ to be 0. This confirms that the minimal surface equation is satisfied, and therefore the mean curvature $H = 0$.