ON SYMPLECTIC GEOMETRY

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Abstract. In Differential Geometry, we are often working with notions of length. In Symplectic Geometry, those very notions are replaced with that of area. This paper aims to give a self contained exposition to key ideas in introductory Symplectic Geometry, including but not limited to Symplectic Forms, Manifolds, and the celebrated Darboux Theorem.

1. Background

Definition 1.1 (Differential Forms)**.** Let M be a smooth manifold. A *differential k-form* on M is a smooth section of the bundle

$$\Lambda^k(T^*M) \to M,$$

that is, a smooth assignment $p \mapsto \omega_p \in \Lambda^k(T_p^*M)$. The space of smooth k-forms is denoted $\Omega^k(M)$. In particular, 0-forms are smooth functions and 1-forms are smooth sections of the cotangent bundle T^*M.

Definition 1.2 (One Forms)**.** In Differential Geometry, we can define a One Form to be a differential form of degree one on a differentiable manifold. It is the mapping of the total space onto a tangent bundle.

This can be thought of as measuring an oriented length.

Definition 1.3 (Two Forms)**.** In Differential Geometry, we can define a Two Form to be a differential form of degree two on a differentiable manifold. It is the mapping of the two dimensional elements onto a tangent space.

This can once more be thought of as measuring an oriented area. We can extend this notion to differntial k-forms, but we will negate that for the purpose of this paper

Definition 1.4 (Exterior Product)**.** If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$ are differential forms, their exterior product is the $(k+\ell)$-form

$$(\omega \wedge \eta)(v_1, \dots, v_{k+\ell}) = \frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\, \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}),$$

for $v_i \in T_pM$. The exterior product is bilinear, associative, and graded-anticommutative:

$$\omega \wedge \eta = (-1)^{k\ell}\, \eta \wedge \omega.$$

In the most natural way, we can combine lower degree forms into higher degree forms

Definition 1.5 (Exterior Derivatives)**.** For $\omega \in \Omega^k(M)$ written in local coordinates as

$$\omega = \sum_I f_I\, dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

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the exterior derivative is the $(k+1)$-form

$$d\omega = \sum_I df_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where $df_I = \sum_j \frac{\partial f_I}{\partial x^j}\, dx^j$.

This simply allows us to take the derivative of forms, telling us how it changes across the manifold

2. Symplectic Manifolds

Having introduced differential forms and their basic operations, we now turn to the special case of 2-forms that are closed and nondegenerate. These give rise to the fundamental objects of our paper, symplectic manifolds.

2.1. **Symplectic Forms.**

2.1.1. *Bilinear Maps.*

Definition 2.1 (Skew-Symmetric Bilinear Maps)**.** Let V be an m-dimensional real vector space, and let

$$\Omega : V \times V \longrightarrow \mathbb{R}$$

be a bilinear map. Ω is skew-symmetric if, for all $u, v \in V$,

$$\Omega(u,v) = -\,\Omega(v,u).$$

Theorem 2.2 (Standard Form)**.** *Let Ω be a skew-symmetric bilinear form on V. Then there exists a basis*

$$u_1, \dots, u_k,\ e_1, \dots, e_n,\ f_1, \dots, f_n$$

of V such that:

$$\Omega(u_i, v) = 0, \quad \text{for all } i \text{ and all } v \in V,$$
$$\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j), \quad \text{for all } i, j,$$
$$\Omega(e_i, f_j) = \delta_{ij}, \quad \text{for all } i, j.$$

Note that for the purposes of this paper, we will forgo the proof of this theorem. An insightful proof and relevant remarks can be found on [dS01]

2.1.2. *Symplectic Vector Spaces and Manifolds.*

Definition 2.3 (Linear Maps)**.** The map $\widetilde{\Omega} : V \to V^*$ is the linear map defined by

$$\widetilde{\Omega}(v)(u) = \Omega(v,u).$$

The kernel of $\widetilde{\Omega}$ is the subspace U introduced above.

Definition 2.4 (Symplectic Maps)**.** A skew-symmetric bilinear form Ω is called symplectic (or nondegenerate if $\widetilde{\Omega}$ is bijective, i.e.,

$$U = \{0\}.$$

Remark 2.5. Nondegeneracy is the essential requirement for a bilinear form to be symplectic. It ensures that the form never "collapses" in any direction. This property is what allows us to identify tangent vectors with covectors, and it lies at the heart of why symplectic structures are so rigid.

Definition 2.6 (Linear Symplectic Structures and Symplectic Vector Spaces)**.** In the prior definition, Ω is referred to as a linear symplectic structure on V, and the pair (V, Ω) is called a symplectic vector space.

Definition 2.7 (Symplectomorphism)**.** A symplectomorphism φ between symplectic vector spaces (V, Ω) and (V', Ω') is a linear isomorphism

$$\varphi : V \xrightarrow{\cong} V'$$

such that

$$\varphi^* \Omega' = \Omega.$$

If such a symplectomorphism exists, (V, Ω) and (V', Ω') are said to be *symplectomorphic*.

Remark 2.8. Symplectomorphisms play the role of "isometries" in symplectic geometry, except they preserve area (given by the symplectic form) instead of lengths.

2.2. **Tautalogical Forms.**

Definition 2.9 (Cotangent Coordinates)**.**

Let $(U, x_1, \ldots, x_n)$ be a coordinate chart for X, with corresponding cotangent coordinates

$$(T^*U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n).$$

Let us define a 2-form ω on T^*U by

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i.$$

To verify that this definition does not depend on the choice of coordinates, consider the 1-form on T^*U given by

$$\alpha = \sum_{i=1}^{n} \xi_i \, dx_i.$$

One can clearly see that

$$\omega = -\, d\alpha.$$

Theorem 2.10. *The 1-form α is intrinsically defined.*

Proof. Let $(U, x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ and $(U', x'_1, \ldots, x'_n, \xi'_1, \ldots, \xi'_n)$ be two cotangent coordinate charts. On $U \cap U'$, the coordinate systems are related by

$$\xi'_j = \sum_i \xi_i \frac{\partial x_i}{\partial x'_j}.$$

Since

$$dx'_j = \sum_i \frac{\partial x'_j}{\partial x_i} \, dx_i,$$

it follows that

$$\alpha = \sum_i \xi_i \, dx_i = \sum_j \xi'_j \, dx'_j = \alpha'.$$

∎

2.2.1. *Tautalogical Forms.* For the purpose of defining Tautalogical Forms and Canonical Symplectic Forms, we will continue with our prior theorem.

Definition 2.11 (Tautalogical Form)**.** We can simply define our Tautalogical Form as the 1-form α

Remark 2.12. The tautological form α is sometimes called the *Liouville 1-form*. It plays the role of a "universal potential" on the cotangent bundle: its exterior derivative $-d\alpha$ produces the canonical symplectic structure. This construction shows that every cotangent bundle comes with a natural symplectic geometry built in.

Definition 2.13 (Canonical Symplectic Form)**.** Similarly, we can simply define our Canonical Symplectic Form as the two form ω

Our Tautalogical and Symplectic Forms may not necessarily mean anything to the reader yet, but they provide valuable definitions.

Let

$$M = T^*X \xrightarrow{\pi} X$$

be the natural projection. A point $p \in M$ can be written as $p = (x, \xi)$, where $\xi \in T_x^*X$.

Definition 2.14 (Tautalogical 1-Form)**.** The tautological 1-form α is defined pointwise by

$$\alpha_p = (d\pi_p)^*\xi \ \in T_p^*M,$$

where $(d\pi_p)^*$ denotes the transpose (pullback) of $d\pi_p$. In other words,

$$(d\pi_p)^*\xi = \xi \circ d\pi_p.$$

Definition 2.15 (Canonical Symplectic 2-Form)**.** The canonical symplectic 2-form ω on T^*X is defined by

$$\omega = -\, d\alpha.$$

In local coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, this takes the form

$$\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i.$$

3. Symplectomorphisms

3.1. **Submanifolds.**

Definition 3.1 (Immersion)**.** Let M and X be smooth manifolds with $\dim X < \dim M$.

A smooth map $i : X \to M$ is called an *immersion* if, for every point $p \in X$, the differential

$$di_p : T_pX \longrightarrow T_{i(p)}M$$

is injective.

An *embedding* is an immersion that is also a homeomorphism onto its image. A *closed embedding* is a proper, injective immersion.

Definition 3.2 (Submanifold)**.** A submanifold of M is a manifold X together with a closed embedding

$$i : X \hookrightarrow M.$$

Definition 3.3 (Conormal Spaces)**.** Let S be a k-dimensional submanifold of an n-dimensional manifold X.

The conormal space at $x \in S$ is

$$N_x^*S = \{\xi \in T_x^*X \ \mid \ \xi(v) = 0 \text{ for all } v \in T_xS\}.$$

Definition 3.4 (Conormal Bundles)**.** The conormal bundle of S is

$$N^*S = \{(x, \xi) \in T^*X \ \mid \ x \in S, \ \xi \in N_x^*S\}.$$

Theorem 3.5. *Let $i : N^*S \hookrightarrow T^*X$ denote the inclusion map, and let α be the tautological 1-form on T^*X. Then*

$$i^*\alpha = 0.$$

Proof. Choose local coordinates $(U, x_1, \dots, x_n)$ on X centered at a point $x \in S$ and adapted to S so that

$$U \cap S = \{x_{k+1} = \cdots = x_n = 0\}.$$

Let $(T^*U, x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ be the corresponding cotangent coordinates. In these coordinates, the submanifold $N^*S \cap T^*U$ is given by

$$x_{k+1} = \cdots = x_n = 0, \qquad \xi_1 = \cdots = \xi_k = 0.$$

Since $\alpha = \sum_{i=1}^n \xi_i \, dx_i$ on T^*U, for $p \in N^*S$ we have

$$(i^*\alpha)_p = \alpha_p\big|_{T_p(N^*S)} = \sum_{i>k} \xi_i \, dx_i \bigg|_{\text{span}\{\frac{\partial}{\partial x_i}, \ i \leq k\}} = 0.$$

■

Definition 3.6 (Twisted Product Form)**.** For manifolds M, N with projections π_M, π_N, forms $\omega \in \Omega^p(M)$, $\eta \in \Omega^q(N)$, and $f \in C^\infty(M \times N)$, the twisted product form is

$$\omega \mathbin{\widetilde{\times}_f} \eta = (f \, \pi_M^*\omega) \wedge \pi_N^*\eta.$$

Remark 3.7. The twisted product construction is a way to combine symplectic forms from two manifolds into a single structure on the product. It foreshadows the importance of product symplectic manifolds in Floer theory and mirror symmetry.

4. Local Forms

Definition 4.1.

- They are **symplectomorphic** if there exists a diffeomorphism $\varphi : M \to M$ such that $\varphi^*\omega_1 = \omega_0$.
- They are **strongly isotopic** if there exists an isotopy $\rho_t : M \to M$ with $\rho_1^*\omega_1 = \omega_0$.
- They are **deformation-equivalent** if there exists a smooth family of symplectic forms $\{\omega_t\}_{t \in [0,1]}$ interpolating between ω_0 and ω_1.
- They are simply called **isotopic** if they are deformation-equivalent and the cohomology class $[\omega_t]$ is independent of t.

Remark 4.2. At this stage, we shift from algebraic constructions to more flexible geometric equivalences between symplectic forms. These notions (symplectomorphic, isotopic, deformation-equivalent) provide the language for comparing symplectic structures, just as diffeomorphisms and isotopies do in topology.

Definition 4.3 (Moser Trick)**.** Let $\{\omega_t\}_{t\in[0,1]}$ be a smooth family of symplectic forms on a manifold M such that $[\omega_t]$ is constant in de Rham cohomology. Then there exists an isotopy $\varphi_t : M \to M$ with $\varphi_t^*\omega_t = \omega_0$.

Note that this is an important theorem with an interesting proof. We defer this to [dS01].

Remark 4.4. The Moser trick is a powerful deformation argument. Its key insight is that instead of working directly with symplectic forms, one studies the vector fields that transport them, thereby constructing isotopies.

Theorem 4.5 (Moser Theorem)**.** *Let M be a compact manifold with symplectic forms ω_0 and ω_1. Suppose that $\{\omega_t\}_{0\le t\le 1}$ is a smooth family of closed 2-forms interpolating between ω_0 and ω_1, satisfying:*

(1) ***Cohomology condition:*** *$[\omega_t]$ is independent of t, i.e.*
$$\frac{d}{dt}[\omega_t] = \left[\frac{d}{dt}\omega_t\right] = 0,$$
(2) ***Nondegeneracy condition:*** *ω_t is nondegenerate for all $t \in [0,1]$.*

Then there exists an isotopy $\rho_t : M \to M$ such that
$$\rho_t^*\omega_t = \omega_0, \quad 0 \le t \le 1.$$

Proof. From the theorem we see:

(1) Since $[\omega_t]$ is constant in cohomology, there exists a family of 1-forms μ_t such that
$$\frac{d}{dt}\omega_t = d\mu_t, \qquad 0 \le t \le 1.$$
A smooth family μ_t can indeed be chosen, for example using the Poincaré lemma for compactly-supported forms together with the Mayer–Vietoris sequence.

(2) By nondegeneracy of ω_t, there exists a unique vector field v_t satisfying the **Moser equation:**
$$\iota_{v_t}\omega_t + \mu_t = 0.$$

Let ρ_t be the isotopy generated by v_t. Then
$$\frac{d}{dt}\big(\rho_t^*\omega_t\big) = \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d}{dt}\omega_t\right) = \rho_t^*(d(\iota_{v_t}\omega_t) + d\mu_t) = 0.$$
Hence $\rho_t^*\omega_t = \omega_0$ for all t. ■

Definition 4.6 (Moser Equation)**.**
$$\iota_{v_t}\omega_t + \mu_t = 0$$

Theorem 4.7 (Moser-Relative Theorem)**.** *Let M be a manifold, X a compact submanifold of M, and $i : X \hookrightarrow M$ the inclusion map. Suppose ω_0 and ω_1 are symplectic forms on M.*

Hypothesis: *For every $p \in X$, we have*
$$\omega_0|_p = \omega_1|_p.$$
Conclusion: *There exist neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that*
$$\varphi^*\omega_1 = \omega_0.$$

Proof. The proof of this theorem is very interesting, but not necessary to the understanding of Darboux Theorem. Thus we defer the reader to [dS01] ■

4.1. **Darboux Theorem.**

Theorem 4.8 (Darboux Theorem)**.** *Let* (M, ω) *be a symplectic manifold and let* $p \in M$*. Then there exists a coordinate chart* $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ *centered at* p *such that, on* U*,*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Proof. We apply the Moser relative theorem (Theorem 7.4) to the case $X = \{p\}$. Choose a symplectic basis of T_pM to construct local coordinates $(x'_1, \ldots, x'_n, y'_1, \ldots, y'_n)$ near p, so that

$$\omega_p = \sum dx'_i \wedge dy'_i \Big|_p.$$

On a neighborhood U', we consider two symplectic forms:

$$\omega_0 = \omega, \qquad \omega_1 = \sum dx'_i \wedge dy'_i.$$

By Moser's theorem, there exist neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of p and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ with

$$\varphi(p) = p, \qquad \varphi^*\Big(\sum dx'_i \wedge dy'_i\Big) = \omega.$$

Hence, setting $x_i = x'_i \circ \varphi$, $y_i = y'_i \circ \varphi$, we obtain the desired coordinates. ■

If in the Relative Moser Theorem we instead assume that X is an n-dimensional submanifold with

$$i^*\omega_0 = i^*\omega_1 = 0,$$

for the inclusion $i : X \hookrightarrow M$, then X is a Lagrangian submanifold for both ω_0 and ω_1. In this case, Weinstein's theorem guarantees that the same conclusion holds, though additional algebraic work is required.

Remark 4.9. This result explains why symplectic geometry has no local curvature invariants, in stark contrast to Riemannian geometry. The richness of symplectic topology is entirely global.

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Bibliography

References

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