The Brachistochrone: Euler–Lagrange, the Cycloid, and Snell's Law

Mihir G.

August 12, 2025

Abstract

Among all frictionless curves in a vertical plane joining a higher point A to a lower point B, the *brachistochrone* is the one that minimizes the descent time under uniform gravity. This paper derives the Euler–Lagrange equation from first principles, formulates the brachistochrone as a least-time variational problem, and solves it via the Beltrami first integral to obtain the cycloid in closed parametric form. Finally, we relate the brachistochrone's first integral to the continuous form of Snell's law for a stratified optical medium, clarifying the equivalence with Fermat's principle of least time.

1 Introduction

The brachistochrone problem, posed by Johann Bernoulli in 1696, asks for the curve of fastest descent from A to B in a uniform gravitational field. The celebrated solution is a *cycloid*, a result that helped crystallize the calculus of variations. Our aim is expository: we (i) derive the Euler–Lagrange equation for a one-dimensional functional; (ii) apply it to the brachistochrone and solve by the Beltrami identity; and (iii) connect the resulting first integral to Snell's law in a vertically stratified medium. We deliberately omit auxiliary topics (e.g., Legendre or Weierstrass checks) to focus on these core elements.

Historically, Bernoulli emphasized the optics analogy: a particle descending under gravity behaves like a light ray in a medium whose refractive index varies with depth, so the least-time curve obeys a refraction law. This observation foreshadows our conclusion that the brachistochrone's conserved quantity is the continuous Snell law for a gradient index.

2 Euler–Lagrange for least time

Let

$$J[y] \;=\; \int_{x_0}^{x_1} L\big(y(x), y'(x), x\big)\, dx$$

with fixed endpoints $y(x_0) = y_0$, $y(x_1) = y_1$. Consider the variation $y_\varepsilon = y + \varepsilon\eta$ where $\eta(x_0) = \eta(x_1) = 0$ and η is smooth. Differentiating at $\varepsilon = 0$ and integrating by parts yields

$$\delta J \;=\; \int_{x_0}^{x_1} \Big(L_y - \frac{d}{dx} L_{y'}\Big)\, \eta(x)\, dx,$$

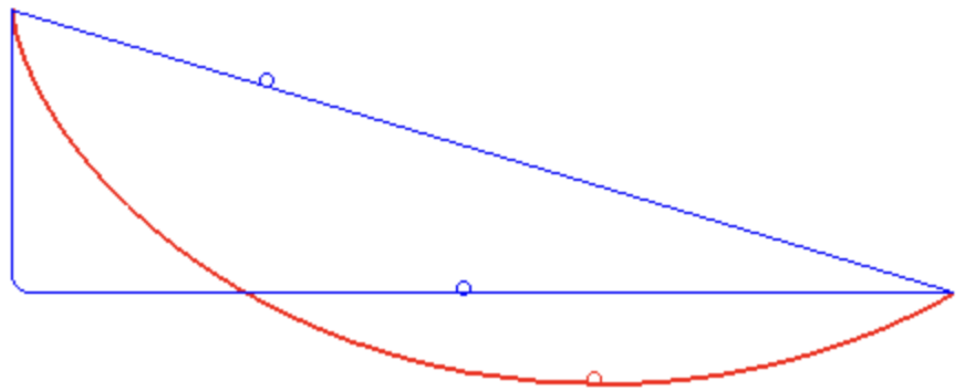

Figure 1: Problem geometry and trial paths.

so stationarity for all η implies the *Euler–Lagrange equation*

$$L_y - \frac{d}{dx} L_{y'} \;=\; 0. \tag{1}$$

When L is independent of x, the *Beltrami identity* (a first integral) follows:

$$L - y' L_{y'} \;=\; \text{constant.} \tag{2}$$

For the brachistochrone, we place $A = (0,0)$ and measure y *downward*. Energy conservation gives the speed $v(y) = \sqrt{2gy}$ (start from rest at A). With $ds = \sqrt{1+y'^2}\,dx$, the travel time is

$$T[y] \;=\; \int \frac{ds}{v} \;=\; \frac{1}{\sqrt{2g}} \int_0^b \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx \;=\; \frac{1}{\sqrt{2g}} \int_0^b L(y,y')\,dx, \qquad L(y,y') = \frac{\sqrt{1+y'^2}}{\sqrt{y}}. \tag{3}$$

The constant prefactor does not affect extremals. Note the vertical tangent at the start is admissible: the integrable singularity $y^{-1/2}$ keeps $T[y]$ finite near $y = 0$.

Beltrami identity: derivation and meaning. Because L has no explicit x-dependence, differentiate $H := L - y' L_{y'}$ along any extremal:

$$\frac{dH}{dx} = L_y y' + L_{y'} y'' - y'' L_{y'} - y' \frac{d}{dx} L_{y'} = y' \Big(L_y - \frac{d}{dx} L_{y'} \Big) = 0,$$

using (1). Hence H is constant, giving

$$\frac{1}{\sqrt{y}\,\sqrt{1+y'^2}} = C > 0 \quad\iff\quad y(1+y'^2) = k, \qquad k = C^{-2}. \tag{4}$$

Writing ϕ for the angle with the horizontal, $\cos\phi = 1/\sqrt{1+y'^2}$ converts (4) to

$$\frac{\cos\phi}{\sqrt{y}} = \text{constant}, \tag{5}$$

a form we will recognize as a continuous Snell law in Section 4.

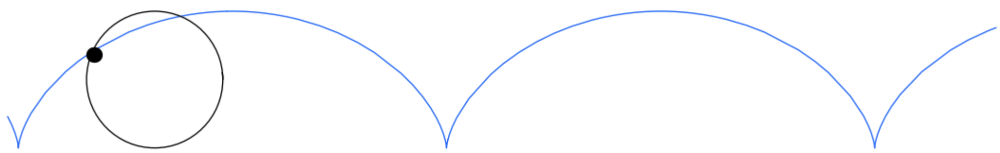

Figure 2: The cycloid solution. A labeled plot helps fix the geometry and the role of θ_B.

Near-cusp integrability at the start. From (4), $y' = \sqrt{(k-y)/y} \sim \sqrt{k/y}$ as $y \to 0^+$. Thus $dx = dy/y' \sim \sqrt{y/k}\,dy$, and the time element satisfies

$$dt = \frac{\sqrt{1+y'^2}}{\sqrt{2g\,y}}\,dx = \frac{1}{\sqrt{2g}} \cdot \frac{\sqrt{k/y}}{\sqrt{y}} \cdot \sqrt{\frac{y}{k}}\,dy = \frac{1}{\sqrt{2g}} \cdot \frac{dy}{\sqrt{y}},$$

which is integrable at $y = 0$. The vertical tangent is therefore admissible and the total travel time remains finite.

A direct application of (1) is possible, but (2) is especially efficient here because L has no explicit x-dependence; it leads immediately to (4), a first integral solvable by quadrature.

3 Solving the brachistochrone: the cycloid

From (4),

$$\frac{dy}{dx} = \sqrt{\frac{k-y}{y}}, \qquad 0 < y < k \quad \Longrightarrow \quad dx = \sqrt{\frac{y}{k-y}}\,dy.$$

Use the standard substitution $y = r(1-\cos\theta)$ with $r = \frac{k}{2}$. Then $dy = r\sin\theta\,d\theta$ and $k-y = r(1+\cos\theta)$, giving

$$dx = r(1-\cos\theta)\,d\theta.$$

Integrating from the cusp ($\theta = 0$) gives the parametric cycloid:

$$x(\theta) = r(\theta - \sin\theta), \qquad y(\theta) = r(1-\cos\theta), \qquad \theta \in [0, \theta_B]. \tag{6}$$

The endpoint conditions $x(\theta_B) = b$ and $y(\theta_B) = \beta$ pick out the unique pair (r, θ_B) consistent with the prescribed $A \to B$ geometry (for $0 < \theta_B \le \pi$ one lands with a horizontal tangent).

A short check confirms that the cycloid indeed satisfies (4):

$$y(\theta)\left(1 + y'(\theta)^2\right) \;=\; r(1-\cos\theta)\left(1 + \cot^2\frac{\theta}{2}\right) \;=\; r(1-\cos\theta)\csc^2\frac{\theta}{2} \;=\; 2r \;=\; k.$$

Thus (6) is the unique extremal compatible with the endpoints, and it minimizes the travel time among nearby curves.

Endpoint fit and uniqueness of (r, θ_B)**.** Set $F(\theta) = \theta - \sin\theta$ and $G(\theta) = 1 - \cos\theta$ for $\theta \in (0, 2\pi)$. Then $F'(\theta) = 1 - \cos\theta > 0$ and $G'(\theta) = \sin\theta \ge 0$ (strictly on $(0,\pi)$), so F is strictly increasing on $(0, 2\pi)$ and G is strictly increasing on $(0, \pi)$. From (6),

$$\frac{b}{r} = F(\theta_B), \qquad \frac{\beta}{r} = G(\theta_B).$$

Given (b, β) with $\beta > 0$, solve $b/\beta = F(\theta_B)/G(\theta_B)$: the ratio is strictly increasing on $(0, \pi)$, so θ_B is unique in $(0, \pi]$; then $r = \beta/G(\theta_B)$ is uniquely determined. This explains precisely how the endpoint data fix the cycloid parameters.

4 Snell's law and the optical analogy

Fermat's principle states that light rays follow paths that extremize optical travel time. For a vertically stratified medium with refractive index $n = n(y)$, the travel-time functional for a graph $y(x)$ reads

$$T[y] \propto \int n(y(x)) \sqrt{1 + y'^2}\, dx.$$

Because the integrand is x-independent, the Beltrami identity yields the conserved quantity

$$\frac{n(y)}{\sqrt{1 + y'^2}} \;=\; \text{constant.} \tag{7}$$

If α is the angle the tangent makes with the horizontal, then $\cos\alpha = 1/\sqrt{1 + y'^2}$, so (7) is

$$n(y)\cos\alpha \;=\; \text{constant,} \tag{8}$$

the continuous (differential) form of Snell's law for a unidirectionally varying index. This is the continuum analogue of $n_1 \sin\theta_1 = n_2 \sin\theta_2$ for discrete layers.

From discrete refraction to the continuous law. Consider a stack of thin horizontal layers with indices $n_0, n_1, \dots, n_N$, and let θ_j be the angle from the normal (vertical) in layer j. Stationarity at each interface gives

$$n_j \sin\theta_j = n_{j+1} \sin\theta_{j+1} \quad \text{for all } j.$$

Hence $n_j \sin\theta_j$ is constant across the stack. Writing $\alpha_j = \frac{\pi}{2} - \theta_j$ for the angle from the horizontal, this is $n_j \cos\alpha_j = \text{const}$. Let the layer thickness $\Delta y \to 0$ with $n_j \to n(y)$ smoothly; then $n(y)\cos\alpha$ is constant along the ray, which is exactly (8). Thus the differential Snell law is the limit of many small refractions.

For the brachistochrone, $v(y) = \sqrt{2gy}$ so an *effective* optical index $n(y) \propto 1/v(y) \propto y^{-1/2}$ makes the light-ray problem identical to the bead's least-time problem. Then (8) becomes

$$\frac{\cos\alpha}{\sqrt{y}} = \text{constant,}$$

which is exactly the first integral (4). Thus the cycloid appears simultaneously as the descent curve and as the refracted ray in a medium with $n(y) \propto y^{-1/2}$.

Concluding remark. The Euler–Lagrange framework, the Beltrami identity, and Fermat's principle fit together seamlessly: a single conserved quantity both integrates the brachistochrone and encodes the continuous Snell law for a stratified medium.

5 Applications and Modern Relevance

The brachistochrone is a classic problem, but the ideas used to solve it are still essential in modern science and engineering. The core concepts show up in surprising places, from mapping our planet to designing robots.

Geophysics and Optics. The link between the brachistochrone and the path of light is key to understanding our own planet. When an earthquake happens, seismic waves travel through the Earth's layers, which have different densities. This causes the waves to curve and follow the fastest possible path, just like the bead on the wire. By tracking the exact time these waves arrive at stations around the world, scientists can work backward to map the inside of the Earth. This same principle is used in *gradient-index (GRIN) optics*, a technology that builds tiny, flat lenses (used in medical scopes, for example) by precisely changing the properties of the material to bend light along an optimal path.

Optimal Control. More broadly, the method used to solve the brachistochrone, the calculus of variations, is the foundation for a whole field called *optimal control theory*. This field is all about finding the best possible way to do something to minimize a cost, like time or fuel. The brachistochrone is the original blueprint for these kinds of problems. Today, engineers use the same fundamental approach to calculate the most fuel-efficient route for a spaceship, program a robot arm to move as quickly as possible, or even model financial strategies.

References

[1] I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Prentice–Hall, 1963; Dover reprint, 2000. (Accessible scan at Internet Archive.) `https://archive.org/details/gelfand-fomin-calculus-of-variations`

[2] P. J. Olver, *Calculus of Variations and Optimal Control*, Univ. of Minnesota lecture notes (2022), `https://www-users.cse.umn.edu/~olver/ln_/cvc.pdf`

[3] E. A. Carlen, *Note on the Brachistochrone Problem*, Rutgers University notes (2013), `https://sites.math.rutgers.edu/~carlen/292S13/brach.pdf`

[4] J. B. Tatum, *Geometric Optics* (LibreTexts), Section 1.8 "Differential Form of Snell's Law", `https://phys.libretexts.org/Bookshelves/Optics/Geometric_Optics_(Tatum)/01:_Reflection_and_Refraction/1.08:_Differential_Form_of_Snell's_Law`

[5] *Gradient-index optics*, Wikipedia (overview and figures for GRIN media), `https://en.wikipedia.org/wiki/Gradient-index_optics`

[6] H. Erlichson, "Johann Bernoulli's brachistochrone solution using Fermat's principle of least time," *European Journal of Physics* **20** (1999) 431–436. (Accessible discussion reprint.) `https://mecheng.iisc.ac.in/suresh/me256/GalileoBP.pdf`

[7] *The Brachistochrone Problem*, MacTutor History of Mathematics Archive (St Andrews), `https://mathshistory.st-andrews.ac.uk/HistTopics/Brachistochrone/`