De Rham Cohomology: The Bridge Between Differential Geometry and Topology

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1 Introduction

De Rham cohomology is an essential tool at the intersection of differential geometry and topology. It allows us to investigate the global properties of smooth manifolds by looking at differential forms and their derivatives. While differential forms come from calculus on manifolds, de Rham cohomology allows us to find valuable topological information, such as how many different types of "holes" are present in a space, without the need for complex algebraic topology methods.

2 Differential Forms and the Exterior Derivative

To define the de Rham cohomology, we begin with differential forms. Let M be a smooth manifold of dimension n.

At each point $p \in M$, we consider the tangent space T_pM, which is the collection of tangent vectors at p. The dual space T_p^*M consists of linear functionals on T_pM (linear maps from $L : T_pM \to \mathbb{R}$). A differential 1-form assigns to each point p an element of T_p^*M in a smooth manner (since $T * M$, the disjoint union of all tangent dual spaces is a smooth manifold the map $\omega : M \to T * M, p \to v* \in T_p * M$ can be checked for smoothness in the usual sense of differentiability of maps between smooth manifolds).

2.1 Multilinearity and Alternating k-maps

A multilinear map is a function $T : V_1 \times \cdots \times V_k \to \mathbb{R}$ that is linear in each argument separately. This means that if all but one input is fixed, then varying the remaining input preserves additivity and scalar multiplication. For example, the dot product is bilinear (linear in each of two arguments), and determinants are multilinear in their column vectors.

An alternating map is a multilinear map $T : V^k \to \mathbb{R}$ with the property that swapping two arguments changes the sign: $T(\dots, v_i, \dots, v_j, \dots) = -T(\dots, v_j, \dots, v_i, \dots)$. As a result, T vanishes whenever two arguments are equal, which means it only depends on linearly independent inputs. The determinant and differential forms are standard examples.

2.2 Differential forms

Higher-degree forms build on this idea: a differential k-form at p is an alternating multilinear map that takes k tangent vectors,

$$\omega_p : \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ times}} \to \mathbb{R},$$

which switches sign when any two arguments are swapped.

When we gather these definitions, a differential k-form ω on M represents a smooth assignment $p \mapsto \omega_p$, where $\omega_p \in \Lambda^k(T_p^*M)$ is the k-th exterior power of the cotangent space. The collection of all smooth differential k-forms on M is represented as $\Omega^k(M)$.

2.3 The Wedge Product $\wedge$

The symbol $\wedge$ represents the wedge product. This operation combines a k-form $\omega \in \Omega^k(M)$ and an ℓ-form $\eta \in \Omega^\ell(M)$ into a $(k+\ell)$-form $\omega \wedge \eta \in \Omega^{k+\ell}(M)$. The wedge product has the following three key properties:

- **Bilinearity:** The wedge product is linear in each argument, $\eta \wedge (\omega + \theta) = \eta \wedge \omega + \eta \wedge \theta$ and $(\eta + \omega) \wedge \theta = \eta \wedge \theta + \omega \wedge \theta$.
- **Antisymmetry:** Swapping two forms changes the sign according to $\eta \wedge \omega = (-1)^{k\ell}\, \omega \wedge \eta$. Specifically, for 1-forms α and β, $\alpha \wedge \beta = -\beta \wedge \alpha$, and $\alpha \wedge \alpha = 0$.
- **Associativity:**
$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta).$$

Example: On $\mathbb{R}^3$ with coordinates (x, y, z),

$$dx \wedge dy = -dy \wedge dx, \quad dx \wedge dx = 0,$$

and $dx \wedge dy \wedge dz$ is a 3-form that represents an oriented volume element.

2.4 k-th exterior power of M

The notation $\wedge^k T_pM$ denotes the k-th exterior power of the tangent space T_pM at a point $p \in M$. It is the vector space of all antisymmetric k-linear combinations of tangent vectors at p. A typical element is of the form

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k, \quad v_i \in T_pM,$$

where the wedge product $\wedge$ is antisymmetric: swapping two vectors changes the sign. This space has dimension $\binom{\dim M}{k}$, and a natural basis is given by $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $\{e_1, \ldots, e_{\dim M}\}$ is a basis of T_pM and $1 \le i_1 < \cdots < i_k \le \dim M$.

The k-th exterior power of the cotangent bundle, $\wedge^k T^*M$, collects these spaces over all points $p \in M$, and its smooth sections are precisely the differential k-forms $\Omega^k(M)$.

2.5 The Exterior Derivative

The main operation on differential forms is the exterior derivative, a linear map

$$d_k : \Omega^k(M) \to \Omega^{k+1}(M),$$

which extends the classical differential of functions. It meets three important properties:

- **Nilpotency:** Applying d twice results in zero, i.e $d_n \circ d_{n+1} = 0$. This means that for any n-form ω, we have $d_{n+1}(d_n(\omega)) = 0$.
- **Compatibility with functions:** For a smooth function $f \in \Omega^0(M) = C^\infty(M)$, df matches the standard differential of f.
- **Leibniz rule:** For $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$,
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

For any k-form ω and any differential form η. In local coordinates $(x^1, \dots, x^n)$, if

$$\omega = \sum_I f_I \, dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

then

$$d\omega = \sum_I df_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where $df_I = \sum_j \frac{\partial f_I}{\partial x^j} dx^j$ is the usual differential of the coefficient function.

One-forms: For a 1-form on $\mathbb{R}^2$, $\omega = P(x,y)\, dx + Q(x,y)\, dy$, the exterior derivative can be written explicitly as

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy.$$

3 De Rham Cohomology

3.1 Exact and closed forms

An exact n-form is an n-form that lies in the image of d_n. A closed n-form is an n-form that lies in the kernel of d_{n+1}, i.e the zero set of d_{n+1} Since $d_{n+1} \circ d_n = 0$, every exact form is also closed.

3.2 What Does This Mean?

Intuitively, closed forms are those that locally resemble derivatives of other forms. Exact forms are those that globally act as derivatives of other forms. The distinction between these two concepts captures global topological information.

3.3 De Rham Cohomology Groups

The de Rham cohomology indicates how every closed form fails to be exact. For each degree k, we define the nth de Rham Cohomology group of M to be the quotient group

$$H^n_{\mathrm{dR}}(M) := \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}} = \frac{\ker(d_{n+1})}{Im(d_n)}.$$

(Since the exterior derivative is a linear map, the kernel of d_{n+1} is a linear subspace of $\Omega^n(M)$ and thus a group under the standard operation of addition. Similar argument applies to the image of d_n)

3.4 Intuition - Holes

Intuitively, de Rham cohomology measures the “holes” in a manifold by distinguishing between closed forms that are exact and those that are not. A closed form that is not exact signals the presence of a hole: there is a global obstruction preventing it from being written as the derivative of another form. For example, a closed 1-form that is not exact on a circle corresponds to the fact that the circle has a loop; similarly, a non-exact 2-form on a 2-sphere detects a two-dimensional “void.” In this way, the dimensions of de Rham cohomology groups—the Betti numbers—count independent holes of different dimensions in the manifold.

3.5 Partitions of Unity

Before we can dive into the concrete examples for de Rham cohomology groups on standard manifolds, we need to introduce the concept of a partition of unity. A partition of unity on a smooth manifold M is a collection of smooth functions $\{\varphi_i\}_{i\in I}$ satisfying the following properties:

- Each function $\varphi_i : M \to [0,1]$ is smooth.
- The collection is locally finite, meaning that for every point $p \in M$, there exists a neighborhood U of p such that all but finitely many φ_i vanish on U.
- The functions sum to one at every point: For every $p \in M$

$$\sum_{i\in I} \varphi_i(p) = 1$$

- Each φ_i has support contained in a specified open set U_i of an open cover $\{U_i\}_{i\in I}$ of M.

Here, the support of a function φ_i, denoted $\text{supp}(\varphi_i)$, is the closure of the set $\{p \in M : \varphi_i(p) \neq 0\}$. In other words, outside the support, the function φ_i is identically zero.

A partition of unity smoothly breaks a manifold into overlapping pieces with weights that add up to one everywhere, letting us combine local information into a global object.

4 Fundamental Theorems and Proof Ideas

In this section, we focus on the theorems that are most relevant for understanding and working with de Rham cohomology. We skip results that are important in geometry and topology, such as Gauss-Bonnet-Chern, Chern-Weil, and full Hodge theory, because they involve more complex machinery than what our knowledge in topology allows us.

Poincaré Lemma

Statement: On any contractible open subset of $\mathbb{R}^n$, every closed k-form with $k \geq 1$ is exact:

$$\text{If} \quad d\omega = 0 \quad \text{then} \quad \omega = d\eta \quad \text{for some } \eta \in \Omega^{k-1}.$$

Proof idea: The key idea is that contractibility ensures there are no "holes" to obstruct exactness. To make this explicit, one defines a homotopy operator K on a star-shaped neighborhood with respect to a fixed base point x_0. For a k-form ω, $K\omega$ is obtained by integrating ω along straight lines from x_0 to each point x, effectively averaging ω along these paths. This operator satisfies the homotopy formula

$$d(K(\omega)) + Kd(\omega) = \text{id},$$

so that for any closed form ω ($d\omega = 0$) we have

$$\omega = d(K\omega),$$

showing that ω is locally exact. Intuitively, K constructs a local primitive by "retracting" the form along straight lines toward the base point, providing an explicit inverse to d in positive degrees.

Stokes' Theorem

Statement: Let M be an oriented smooth manifold of dimension n with a smooth boundary ∂M. If $\omega \in \Omega^{n-1}(M)$, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

Stokes' theorem generalizes several classical results:

- The Fundamental Theorem of Calculus ($n = 1$).
- Green's Theorem and Kelvin-Stokes Theorem ($n = 2$).
- The Divergence Theorem ($n = 3$).

It connects local information ($d\omega$) to global behavior (integrals over boundaries).

Proof idea (elaborated):

1. Cover the manifold M with coordinate charts (U, ϕ) such that $U \subset M$ is diffeomorphic to an open subset of $\mathbb{R}^n$. In these coordinates, differential forms can be expressed using the standard Euclidean basis $dx^1, \ldots, dx^n$, and Stokes' theorem reduces to the classical divergence theorem. This establishes the result locally.
2. Use a smooth partition of unity $\{\varphi_i\}$ subordinate to the cover. Each $\varphi_i \omega$ has support in a single chart, so Stokes' theorem applies to each piece individually.
3. The manifold M is oriented and the charts are oriented compatibly. This ensures that the contributions from overlapping charts combine correctly, so the local results sum to give a global statement over M.
4. Both the exterior derivative d and integration are linear. For $\omega = \omega_1 + \omega_2$,

$$d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2, \qquad \int_{\partial M} (\omega_1 + \omega_2) = \int_{\partial M} \omega_1 + \int_{\partial M} \omega_2.$$

   This allows us to extend Stokes' theorem from each piece $\varphi_i \omega$ to the full form ω.

The theorem is valid for any ω of degree $n - 1$, and by linearity, also holds for sums of such forms.

Connection to de Rham cohomology: If M has no boundary, Stokes' theorem states

$$\int_M d\omega = 0,$$

indicating that the integral over M relies only on the cohomology class of ω. This is why integrals of closed forms over cycles define invariants in de Rham cohomology.

De Rham's Theorem

Statement: For any smooth manifold M, the de Rham cohomology is naturally isomorphic to singular cohomology with real coefficients:

$$H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{sing}}(M; \mathbb{R}).$$

Proof idea: One defines a map from differential forms to singular cochains by integrating over simplices:

$$\omega \mapsto \left(\sigma \mapsto \int_\sigma \omega \right).$$

Using tools such as the Mayer-Vietoris sequence (A tool which relates the cohomology of a space to that of two overlapping parts, helping compute cohomology by breaking the space into simpler pieces, similar to how the Seifert van Kampen theorem does the same thing for fundamental groups of connected spaces) and partitions of unity, one shows this map induces an isomorphism (i.e a bijective homomorphism) between the two groups . This theorem guarantees that the analytical approach of differential forms fully retrieves the classical topological invariants represented by singular cohomology. De Rham cohomology connects analysis, geometry, and topology by measuring global aspects of manifolds through differential forms.

5 Examples and Explicit Computations

In this section we will look at and compute the de Rham cohomology groups of a few standard

5.1 Euclidean Space $\mathbb{R}^n$

Consider $M = \mathbb{R}^n$ with coordinates $(x_1, \ldots, x_n)$.

0th de Rham cohomology group Since a 0-form on any manifold is a smooth assignment of a scalar to every point on the manifold (as opposed to - for example - a one-form, which assigns a covector to each point on the manifold in a smooth manner), we get that

$$\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n),$$

which corresponds to infinitely differentiable functions. The exterior derivative acts as the usual differential,

$$d_0 : \Omega^0 \to \Omega^1, \quad f \mapsto df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$$

A 0-form f is closed if and only if $df = 0$, which indicates that f is constant.

Exact 0-forms do not exist since $d : \Omega^{-1} \to \Omega^0$ is undefined. Thus, $H^0_{\mathrm{dR}}(\mathbb{R}^n) = \{\text{constant functions}\} \cong \mathbb{R}$.

Higher-degree forms The Poincaré Lemma states that on $\mathbb{R}^n$, every closed k-form with $k \geq 1$ is exact.

Specifically, if $\omega \in \Omega^k(\mathbb{R}^n)$ satisfies $d\omega = 0$, then there is an $\eta \in \Omega^{k-1}(\mathbb{R}^n)$ such that $\omega = d\eta$. Thus, $H^k_{\mathrm{dR}}(\mathbb{R}^n) = 0 \quad \text{for all } k \geq 1$.

Example: A closed 1-form that is exact Consider the 1-form

$$\omega = 2x\,dx + 2y\,dy \quad \text{on } \mathbb{R}^2.$$

To check if ω is closed, compute its exterior derivative:

$$d\omega = d(2x\,dx + 2y\,dy) = d(2x) \wedge dx + d(2y) \wedge dy.$$

Since

$$d(2x) = 2\,dx, \quad d(2y) = 2\,dy,$$

we have

$$d\omega = 2\,dx \wedge dx + 2\,dy \wedge dy = 0 + 0 = 0,$$

because $dx \wedge dx = dy \wedge dy = 0$. Hence, ω is closed.

Now, we find a function f such that $df = \omega$:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 2x\,dx + 2y\,dy.$$

By integrating,

$$f(x, y) = x^2 + y^2 + C.$$

Thus, $\omega = df$ is exact.

5.2 The Sphere S^n

The n-sphere can be represented in $\mathbb{R}^{n+1}$ as $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

0th de Rham cohomology group Since S^n is connected, the 0th de Rham cohomology group is

$$H^0_{\mathrm{dR}}(S^n) \cong \mathbb{R}.$$

0-forms are simply smooth functions, and a 0-form f is closed if $df = 0$, which means f is constant. A constant function cannot be written as the derivative of another globally defined function on S^n, so it is non-exact. Therefore, every closed 0-form on S^n is a real multiple of the constant functions, and the space of 0th cohomology classes is one-dimensional, isomorphic to $\mathbb{R}$.

Volume Forms The n-sphere $S^n \subset \mathbb{R}^{n+1}$ has a natural volume form ω_{S^n}, which is an n-form giving the standard surface measure on the sphere. It can be obtained by restricting the Euclidean volume form in $\mathbb{R}^{n+1}$ to the tangent space of S^n at each point. Intuitively, this volume form measures“surface area“ on the 2-sphere/the ”volume” of the 3-sphere and is not exact

Example: For the 2-sphere $S^2 \subset \mathbb{R}^3$ with spherical coordinates (θ, ϕ), the natural volume form is

$$\omega_{S^2} = \sin\theta \, d\theta \wedge d\phi,$$

which corresponds to the usual surface area element. Integrating this form over S^2 gives the total area 4π.

To demonstrate this clearly:

- Assume $\omega_{S^n} = d\eta$ for some $\eta \in \Omega^{n-1}(S^n)$. - By Stokes' theorem,

$$\int_{S^n} \omega_{S^n} = \int_{S^n} d\eta = \int_{\partial S^n} \eta = 0,$$

because S^n has no boundary, $\partial S^n = \varnothing$.

- However, the volume form integrates to a nonzero volume for the sphere, which leads to a contradiction.

Thus, ω_{S^n} is closed but not exact.

kth De Rham Cohomology Group of S^n Let S^n be the n-dimensional sphere. Its de Rham cohomology is

$$H^k_{\mathrm{dR}}(S^n) \cong \begin{cases} \mathbb{R}, & k = 0 \text{ or } k = n, \\ 0, & 0 < k < n. \end{cases}$$

- $H^0_{\mathrm{dR}}(S^n)$ consists of constant functions, so $H^0_{\mathrm{dR}}(S^n) \cong \mathbb{R}$.
- For $0 < k < n$, every closed k-form is exact due to the contractibility of S^n minus a point, so $H^k_{\mathrm{dR}}(S^n) = 0$.
- For $k = n$, the volume form ω_n is closed but not exact (its integral over S^n is nonzero), giving $H^n_{\mathrm{dR}}(S^n) \cong \mathbb{R}$.

Cohomology in intermediate degrees

For $0 < k < n$, the cohomology groups are zero: $H^k_{\mathrm{dR}}(S^n) = 0$. This indicates there are no nontrivial ”holes” of intermediate dimension.

5.3 The Torus $T^n = (S^1)^n$

The n-torus is the cartesian product of n copies of the circle S^1:

$$T^n = S^1 \times \cdots \times S^1.$$

Each circle S^1 can be described by an angular coordinate $\theta_i \in [0, 2\pi)$. On each factor, the 1-form $d\theta_i$ is closed but not exact, because θ_i is not a globally defined function on the circle (it jumps by 2π at the boundary).

Cohomology groups The de Rham cohomology of the torus is generated by wedge products of the $d\theta_i$, meaning that every cohomology class of degree k can be represented as a linear combination of wedge products of k of the basic 1-forms $d\theta_1, \ldots, d\theta_n$, with indices increasing: $d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_k}$, $1 \leq i_1 < \cdots < i_k \leq n$. In particular, the $d\theta_i$ form a basis of $H^1(T^n)$, and all higher-degree cohomology classes are constructed from these through the wedge product. Thus :

$$H^k_{\mathrm{dR}}(T^n) = \bigwedge^k \mathbb{R}^n,$$

with basis elements $\{d\theta_{i_1} \wedge d\theta_{i_2} \wedge \cdots \wedge d\theta_{i_k} | 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$.

Example for T^2 Consider the 2-torus $T^2 = S^1 \times S^1$ with angular coordinates θ_1 and θ_2. The 1-forms $d\theta_1$ and $d\theta_2$ are closed but not exact. Hence, the de Rham cohomology groups are

$$H^0_{\mathrm{dR}}(T^2) \cong \mathbb{R}, \quad H^1_{\mathrm{dR}}(T^2) \cong \mathbb{R}^2, \quad H^2_{\mathrm{dR}}(T^2) \cong \mathbb{R}.$$

The 2-form $d\theta_1 \wedge d\theta_2$ is closed and represents the fundamental cohomology class in degree 2.

Non-exactness of $d\theta_i$: Suppose $d\theta_1 = df$ for some smooth function f on T^2. Then integrating along the loop $\theta_1 \mapsto \theta_1 + 2\pi$ gives

$$\int_0^{2\pi} d\theta_1 = f(\theta_1 + 2\pi, \theta_2) - f(\theta_1, \theta_2) = 2\pi,$$

which is impossible for a well-defined function f. Therefore $d\theta_1$ is not exact. A similar argument shows $d\theta_2$ is not exact.

6 Citations

6.1 Literature:

- Intro to smooth manifolds by John M. Lee
- Differential Forms in Algebraic Topology by Raoul Bott

6.2 Proofs:

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