LIE GROUPS AND LIE ALGEBRAS

MARIGOLD STRUPP

1. Manifolds

The fundamental goal of Lie theory is to understand smooth, continuous symmetry. In an elementary group theory course, one often encounters the cyclic groups, the groups consisting of rotations of regular n-gons. These groups tend to feel very discrete, words like smooth and continuous do not describe them. Another common group is $SO_2(\mathbb{R})$, the space of rotations in two dimensional (euclidean) space, or more generally $SO_n(\mathbb{R})$, the space of rotations in n-dimensional space. These, unlike the cyclic groups feel smooth and continuous. In order to mathematically articulate this feeling we need to understand what it means for an object to be smooth.

For 1 and 2 dimensional objects we already have such a notion. A 1 dimensional smooth object is a smooth curve and a 2 dimensional smooth object is a smooth surface. A curve can be thought of as something that locally looks like $\mathbb{R}$ and similarly a surface can be thought of as something that looks locally like $\mathbb{R}^2$. So, we might define a smooth object in n-dimensions to be a topological space that looks locally like $\mathbb{R}^n$. This definition has a problem, it only captures part of the picture, under this definition a square, which has sharp corners and is definitely not smooth, would be smooth. The issue here is that a smooth object must have more structure than just a topology, it has to have what is called a smooth manifold structure. With that in mind we have the following definition

Definition 1.1 (Smooth manifolds)**.** An n dimensional smooth manifold is a topological space that looks locally like $\mathbb{R}^n$, in the sense that every point has an open neighborhood homeomorphic to $\mathbb{R}^n$. Equipped with a maximal choice of open cover $\{U_i\}$ such that for each U_i we have a homeomorphism (called a chart) $\varphi_i : U_i \to \mathbb{R}^n$ where for any U_i, U_j, $\varphi_i|_{U_i \cap U_j} \circ \varphi_j^{-1}|_{U_i \cap U_j}$ is smooth.

Example. Along with curves and surfaces an easy example of a smooth manifold is an n sphere. Here the charts are given by the stereographic projection from the north pole and the stereographic projection from the south pole

Example. Any open subset U of $\mathbb{R}^n$ is a manifold because if $p \in U$ there exists some epsilon ball $p \in B \subset U$. Since epsilon balls are smoothly homeomorphic to $\mathbb{R}^n$ the manifold structure follows.

There are two important things to note about this definiton. The first is that a smooth manifold is an additional structure on top of a topology, meaning one could have two identical topological spaces with vastly different smooth manifold structures. The second is that nowhere does this definition require that a manifold embed into euclidean space. While it is true that every manifold can be embedded in euclidean space (This fact is known as the weak Whitney embedding theorem [2]), the proof is highly non-trivial and so it is best to think of manifolds as structures unto themselves as opposed to subsets of some ambient space.

As with many other mathematical objects, we can not fully talk about smooth manifolds without talking about the functions between them.

Definition 1.2 (Smooth maps)**.** Let M and N be smooth manifolds. Then a function $f : M \to N$ is said to be smooth if for every chart φ of M and every chart ϕ of N the function $\phi \circ f \circ \varphi^{-1}$ is smooth. A smooth bijection with smooth inverse is called a diffeomorphism.

The key class of smooth functions on a manifold M are those from $M \to \mathbb{R}$. In fact, there is an alternate way of defining manifolds by defining it as a topological space M equipped with something called a structure sheaf of continuous functions $M \to \mathbb{R}$ (if you have studied sheaves trying to figure what the conditions on this sheaf are that allow it to define a manifold is a fun exercise.) For our purposes though, the primary reason we care about smooth maps from $M \to \mathbb{R}$ is because such maps have the structure of an associative algebra.

Proposition 1.3. *Let M be a smooth manifold and $C^\infty(M)$ denote the set of smooth functions $f : M \to \mathbb{R}$. Then $C^\infty(M)$ is an associative algebra under the pointwise sum and product.*

Proof. Let $f, g \in C^\infty(M)$, then because the pointwise sum and product of real valued functions already satisfy the ring axioms it is sufficient to show that $fg, f + g \in C^\infty(M)$. Note that a function $h : M \to \mathbb{R}$ is smooth if and only if for every chart ϕ on M, $h \circ \phi^{-1}$ is infinitely differentiable. Then $(fg) \circ \phi^{-1} = (f \circ \phi^{-1})(g \circ \phi^{-1})$ which since the product of two functions is infinitely differentiable, is thus infinitely differentiable. Similarly because the sum of two infinitely differentiable functions $(f + g) \circ \phi^{-1} = (f \circ \phi^{-1}) + (g \circ \phi^{-1})$. Thus if f, g are smooth, $f + g$ and fg are smooth. ∎

This allows us to define one of the key constructions on manifolds, the tangent space. For curves and surfaces the tangent space is the tangent line and tangent plane respectively. Elements of the tangent space are usually thought of as vectors in whatever the ambient space the curve or surface is embedded in. For manifolds however, we have to consider a more intrinsic notion of tangent vector, the question then becomes, what is it that makes a tangent vector a tangent vector? In differential geometry a tangent vector is often used as something we can take a derivative in the direction of. With this in mind we define the tangent space as follows.

Definition 1.4 (The tangent space)**.** For any smooth manifold M and any point $p \in M$, a derivation D is a linear map $C^\infty(M) \to \mathbb{R}$ such that for any $f, g \in C^\infty(M)$

$$D(fg) = D(f)g(p) + f(p)D(g)$$

The tangent space of M at p, denoted T_pM is the space of all derivations. importantly the tangent space is a vector space under the operations

$$(aD)(f) := a(D(f))$$
$$(D_1 + D_2)(f) = D_1(f) + D_2(f)$$

Notably, if c is a constant then for any derivation D we have

$$D(c) = D(1c) = D(1)c + D(c)$$

meaning $D(1) = 0$ however $0 = cD(1) = D(c)$ So derivations sends constants to 0.

Now that we have the notion of a tangent space we should be able to define the derivative of a smooth map. When working with curves and surfaces the derivative is thought of as giving the best linear approximation near a given point. This tells us that the derivative of a smooth function at a point should be a linear map and should thus be a map of the tangent space. Since we are considering tangent vectors as essentially directional derivatives we have for any smooth function a very natural map between the tangent spaces. We will define the derivative as that map.

Definition 1.5 (Total derivative)**.** Suppose $f : M \to N$ is a smooth map and $p \in M$. Then the total derivative or differential at p, denoted $d_p f$ is the map $T_pM \to T_{f(p)}(N)$ given by:

$$(d_p f(v))(g) := v(g \circ f)$$

This is a well defined linear map because

$$\begin{aligned}(d_p f(v))(gh) &= v((gh) \circ f)\\ &= v((g \circ f)(h \circ f))\\ &= v(g \circ f)(h \circ f)(p) + (g \circ f)(p)v(h \circ f)\\ &= (d_p f(v))(g)h(f(p)) + (d_p f(v))(h)g(f(p)\end{aligned}$$

$$\begin{aligned}(d_p f(v))(g + h) &= v((g + h) \circ f)\\ &= v(g \circ f + h \circ f)\\ &= v(g \circ f) + v(h \circ f)\\ &= (d_p f(v))(g) + (d_p f(v))(h)\end{aligned}$$

$$\begin{aligned}d_p f(v + w)(g) &= (v + w)(g \circ f)\\ &= v(g \circ f) + w(g \circ f)\\ &= d_p(f(v))(g) + d_p(f(w))(g)\end{aligned}$$

$$d_p f(rv) = rv(g \circ f) = rd_p f(v)$$

For derivatives in $\mathbb{R}^n$ we have the chain rule. One would hope that it generalizes to manifolds and in fact it does, interestingly however it generalizes to a statement about the functoriality of the derivative.

Proposition 1.6 (The chain rule)**.** *The map taking a (pointed) manifold to its tangent space and a smooth function to its derivative is a functor.*

Proof. First, let $\gamma \in C^\infty(L)$ and $v \in T_pM$. Then

$$(d_p(g \circ f)v)(\gamma) = v(\gamma \circ g \circ f) = ((d_{f(p)}g \circ d_p f)v)(\gamma)$$

And if $\rho \in C^\infty(M)$ we have

$$(d_p\mathrm{id}v)(\rho) = v(\rho \circ \mathrm{id}) = v(\rho)$$

■

The tangent space allows us to define 3 special types of smooth map. These are essentially local versions embeddings, quotient maps, and diffeomorphisms respectively.

Definition 1.7. A smooth map $f : M \to N$ is a

- Immersion if and only if the differential is injective at every point
- Submersion if and only if the differential is surjective at every point
- local diffeomorphism if and only if the differential is an isomorphism at every point

Notably, by something called the inverse function theorem [2] (which we will not prove) f is a local diffeomorphism if and only if for every point $p \in M$ there exists some open $p \in U \subset M$ such that $f|_U$ is a diffeomorphism onto its image. In fact, the inverse function theorem is slightly stronger, it states that the differential of f at p is an isomorphism if and only if there exists some $p \in U \subset M$ such that $f|_U$ is a diffeomorphism onto its image.

Immersions are of particular interest to us because the image of an immersion is known as an immersed submanifold. This is in contrast to what's called an embedded submanifold which is the image of a smooth topological embedding.

One concept that is ubiquitous when dealing with curves and surfaces is the idea of a velocity vector. We are now equipped to define these, they essentially allow us to speak of the properties of curves on a manifold.

Definition 1.8. Let $\gamma : \mathbb{R} \to M$ be a smooth curve on M. At any $t \in \mathbb{R}$ Let 1 be the tangent vector given by

$$1(f) = f'(t)$$

Then the velocity vector of γ at t, is given by $\gamma'(t) := (d_t\gamma)(1)$.

Now that we have defined the tangent space and a few constructions using it we should take a step back and try to understand the structure of the tangent space. This has a surprisingly nice answer, in order to reach that answer we first characterize the tangent space of $\mathbb{R}^n$.

Lemma 1.9. *The tangent space at any point in $\mathbb{R}^n$ is $\mathbb{R}^n$.*

Proof. Suppose $p = (p_1, \dots, p_n) \in \mathbb{R}^n$. Then we have an isomorphism $T_0\mathbb{R}^n \to T_p\mathbb{R}^n$ given by $v \mapsto v_p$ where, letting $f_p = f(x+p)$, we define $v_p(f) := v(f_p)$. This is clearly linear because $(av+w)_p(f) = av(f_p)+w(f_p) = av_p(f)+w_p(f)$ and has an inverse given by $v \in T_p\mathbb{R}^n \mapsto v_{-p}$ where $v_{-p}(f) = v(f_{-p})$.

Then letting ∂^i be the derivation $\partial^i(f) = \frac{\partial f}{\partial x_i}(0)$, it is clear by definition of partial derivatives that the set of ∂^i is linearly independent. Now suppose v is a derivation at 0 and $f \in C^\infty(\mathbb{R}^n)$. Then by the fundamental theorem of calculus

$$f(x_1, \dots, x_n) = f(0) + \int_0^1 \frac{d}{dt} f(tx_1, \dots, tx_n)dt$$

rewriting using the chain rule we have

$$f(x) = f(0) + \sum_{i=1}^n x_i \int_0^1 \partial^i f(tx)dt$$

So

$$v(f) = v(f(0) + \sum_{i=1}^n x_i \int_0^1 \partial^i f(tx)dt) = \sum_{i=1}^n v(x_i)\partial^i f + \sum_{i=1}^n 0v(\int_0^1 \partial^i f(tx)dt) = \sum_{i=1}^n v(x_i)\partial^i f$$

So $\{\partial^i\}$ form a basis for the tangent space meaning the tangent space is isomorphic to $\mathbb{R}^n$. ∎

This immediately characterizes the tangent space of all manifolds.

Proposition 1.10. *Let M be manifold of dimension n, then every tangent space on M is isomorphic to $\mathbb{R}^n$*

Proof. Suppose $p \in M$ and ϕ a chart containing p. Define a map $\phi_* : T_pM \to T_{\phi(p)}\mathbb{R}^n$ by $\phi_* v(f) = v(f \circ \phi^{-1})$. Then this map is linear because

$$\phi_*(av + w)(f) = av(f \circ \phi^{-1}) + w(f \circ \phi^{-1}) = a\phi_* v(f) + \phi_* w(f)$$

and it has an inverse given by $\phi_*^{-1} v(f) = v(f \circ \phi)$. So $T_pM \cong T_{\phi(p)}\mathbb{R}^n \cong \mathbb{R}^n$ ∎

Finally, we will discuss the product of two manifolds, this generalizes the product of topological spaces to manifolds, it will also be necessary to define Lie groups.

Definition 1.11 (Product of manifolds)**.** Suppose M, N are manifolds. Then their product $M \times N$ is given the product topology. This means that we may define charts on $M \times N$ by taking any any chart $\phi_M : U \to \mathbb{R}^m$ on M and any chart $\phi_N : V \to \mathbb{R}^n$ on N and defining a new chart on $M \times N$ by taking $\phi_M \times \phi_N : U \times V \to \mathbb{R}^{n+m}$. So $M \times N$ is an $m + n$ dimensional manifold. One important consequence of this is that the tangent space at any point (a, b) is equal to $T_aM \oplus T_bN$. Explicitely this isomorphism defines (v, w) to be the derivation $(v, w)(\gamma) = v(\gamma|_a) + w(\gamma|_b)$ where $\gamma|_a(m) = \gamma(m, b)$ and $\gamma|_b(n) = \gamma(a, n)$.

It is finally time to talk about Lie groups, these are groups that capture smooth symmetry, they are both groups and manifolds.

Definition 1.12. A Lie group G has both the structure of a group and a smooth manifold, such that group multiplication map $m : G \times G \to G$ is smooth.

As stated previously, you can never really talk about mathematical objects without talking about maps between them.

Definition 1.13. A homomorphism of Lie Groups is a homomorphism of groups that is smooth. An isomorphism of Lie groups is a group isomorphism with smooth inverse

Example. The canonical examples of Lie groups are the matrix groups

- $\mathrm{GL}_n(\mathbb{R})$ is the group of $n \times n$ invertible real matrices
- $\mathrm{SL}_n(\mathbb{R})$ is the group of $n \times n$ matrices with determinant 1
- $\mathrm{O}_n(\mathbb{R})$ is the group of all $n \times n$ matrices with $X^T X = I$
- $\mathrm{SO}_n(\mathbb{R})$ is the group of all $n \times n$ matrices with $X^T X = I$ with determinant 1

Example. The non zero quaternions under multiplication form a lie group

I will leave the above two examples for you to verify if you want to, these verifications are largely computational and generally working through them gives you a good idea of what manifolds and Lie groups feel like.

2. Vector fields and their Lie algebra

An important idea in Lie theory is that we should try to study an algebraic structure on the tangent space (called a Lie algebra) induced by the group multiplication. In order to construct such an algebraic structure we have to look at what are called vector fields. A vector field is roughly a smooth choice of tangent vectors at each point on a manifold. In order to make this rigorous we need to introduce whats called the tangent bundle

Definition 2.1. Let M be a manifold, the tangent bundle of M, TM consists of ordered pairs (p, v) with $p \in M$ and $v \in T_pM$. The natural projection map $\pi : TM \to M$ is given by $\pi : (p, v) \mapsto p$.

This can be given a smooth structure, but that is not necessary for our purposes. We actually need very little to define a smooth vector field.

Definition 2.2 (Vector field)**.** A smooth vector field X is a function $X : M \to TM$ such that

- For any point $p \in M$, denoting the application of X to p by X_p, we have $\pi(X_p) = p$.
- For any $f \in C^\infty(M)$, viewing X_p as a derivation, the function $Xf(p) := X_p(f)$ is smooth.

Notably vector fields form a real vector space under the operations

- $(X + Y)_p = X_p + Y_p$
- $(aX)_p = aX_p$

A nice way of thinking about vector fields is as generalized derivatives. Specifically if $f \in C^\infty(M)$ then we may obtain a new smooth function $X(f)$ called the derivative of f along X by taking $X(f)(p) = X_p(f)$. Then this seems to give natural algebraic structure to vector fields. We can define the composition of two vector fields XY such that $(XY)_p(f) = X_p(Y(f))$. This looks great on the surface, but we run into an issue. namely that this is not, in general, a tangent vector. It doesn't satisfy the product rule because:

$$\begin{aligned} X(Y(fg)) &= X(Y(f)g + fY(g)) \\ &= X(Y(f)g) + X(fY(g)) \\ &= X(Y(f))g + Y(f)X(g) + X(f)Y(g) + fX(Y((g)) \end{aligned}$$

You might notice that this almost satisfies the product rule. $XY(fg) = fXY(g) + XY(f)g$ plus some other stuff. What the you might also nptice is that this other stuff is the same for XY and YX. So, let's define a new algebraic operation, called the Lie bracket, on vector fields by $[X, Y] = XY - YX$. It's easy to check that this is actually a vector field.

Proposition 2.3. *If X, Y are vector fields so is $[X, Y] = XY - YX$.*

Proof. We need only show that this acts linearly on $f, g \in C^\infty(M)$ and that it satisifies the product rule. This is fairly easy, linearity follows because:

$$\begin{aligned} [X, Y](af + g) &= X(Y(af + g)) - Y(X(af + g)) \\ &= aX(Y(f)) + X(Y(g)) - aY(X(f)) - Y(X(g)) \\ &= a(X(Y(f)) - Y(X(f))) + X(Y(g)) - Y(X(g)) \\ &= a[X, Y](f) + [X, Y](g) \end{aligned}$$

and the product rule follows because:

$$\begin{aligned} [X, Y](fg) &= XY(fg) - YX(fg) \\ &= X(Y(f))g + Y(f)X(g) + X(f)Y(g) + fX(Y((g)) \\ &- (Y(X(f))g + Y(f)X(g) + X(f)Y(g) + fY(X((g))) \\ &= (X(Y(f)) - Y(X(f)))g + f(X(Y(g)) - Y(X(g))) \\ &= [X, Y](f)g + f[X, Y](g) \end{aligned}$$

■

There are 3 identities satisfied in general by the Lie bracket. These allow us to decipher all of the general algebraic properties of the Lie bracket.

Proposition 2.4. *Let X, Y, Z be vector fields. Then we have (i) The alternating property* $[X, X] = 0$

(ii) Bilinearity

(iii) The jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

Proof. (i) This is nearly trivial because

$$[X, X] = XX - XX = 0$$

(ii) let $a \in \mathbb{R}$ be a scalar, then

$$[aX+Z, Y] = aX(Y)+Z(Y)-Y(aX+Z) = aX(Y)-aY(X)+Z(Y)-Y(Z) = a[X, Y]+[Z, Y]$$

meaning the Lie bracket is linear in the first slot and it is linear in the second slot because

$$[X, aY+Z] = X(aY+Z)-aY(X)-Z(X) = aX(Y)-aY(X)+X(Z)-Z(X) = a[X, Y]+[X, Z]$$

(iii) The jacobi identity is the hardest to prove of any of these identity but still not terrible. It follows by

$$\begin{aligned}[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] &= XYZ - XZY - YZX + ZYX \\ &+ ZXY - ZYX - XYZ + YXZ \\ &+ YZX - YXZ - ZXY + XZY \\ &= (XYZ - XYZ) + (XZY - XZY) \\ &+ (YZX - YZX) + (ZYX - ZYX) \\ &+ (ZXY - ZXY) + (YXZ - YXZ) \\ &= 0\end{aligned}$$

■

As you might notice, none of these properties depend on the objects operated on by the Lie bracket being vector fields. We only need them to be vectors (or elements of a module but we won't get into that). So we may define a new sort of algebraic structure known as a Lie algebra. Lie algebras generalize the Lie bracket on vector fields and ultimately will be what allows us to "flatten" a Lie group to its tangent space.

Definition 2.5. A Lie algebra $\mathfrak{g}$ is a vector space equipped with a bilinear operation $[-, -]$ called the Lie bracket satisfying

- The alternating property
$$[x, x] = 0$$
- The jacobi identity
$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]]$$

A morphism of Lie algebras is just a linear function between Lie algebras that preserves the Lie bracket.

While it is true that vector fields induce Lie algebras, there is one Lie algebra in particular that I am almost certain you have seen before.

Example. the vector space $\mathbb{R}^3$ is a Lie algebra with the cross product as the Lie bracket. Bilinearity and the alternating property are immediate from the definition, the jacobi identity follows because

$$\begin{aligned} a \times (b \times c) + c \times (a \times b) + b \times (c \times a) &= (a \cdot c)b - (a \cdot b)c \\ &+ (c \cdot b)a - (c \cdot a)b \\ &+ (b \cdot a)c - (b \cdot c)a \\ &= 0 \end{aligned}$$

3. The Lie algebra of a Lie group

The primary goal of this section is to show how exactly we flatten a Lie group to a Lie algebra. The first step to this process is to note that Lie groups act in a very natural way on tangent vectors.

Definition 3.1. Let G be a Lie group and $g, h \in G$. Then denote left multiplication by g on h by $L_g(h)$. Then L_g is smooth by definition and thus we call have the derivative map $d_h L_g : T_h G \to T_{gh} G$ left translation by g.

In particular we are able to talk about those vector fields invariant under this action. These vector fields will turn out to be the key in defining the lie algebra.

Definition 3.2. A smooth vector field X on a Lie group G is said to be left-invariant if for any $g, h \in G$, $d_h L_g(X_h) = X_{gh}$.

Importantly, every left-invariant vector field is determined by its value at 1. This is because for any $g \in G$ we must have $X_g = X_{g1} = (d_1 L_g)(X_1)$. Further if $v \in T_1 G$ then we may define a vector field X^v by $X^v_g = (d_1 L_g)(v)$. We will now show that left-invariant vector fields are closed under the Lie bracket operation because if they are, we would be able to define a Lie algebra structure on the tangent space.

Lemma 3.3. *If X, Y are left-invariant vector fields then $[X, Y]$ is left-invariant.*

Proof. Let G be a Lie group and X, Y be left invariant vector fields on G. Then for any $g, h \in G$ we have

$$((d_h L_g)[X, Y]_h)(f) = [X, Y]_h(f \circ L_g) = X_h(Y(f \circ L_g)) - Y_h(X(f \circ L_g))$$

Then since X is a left invariant vector field

$$(X(f \circ L_g))(h) = X_h(f \circ L_g) = X_{gh}((f) = X(f) \circ L_g)(h)$$

So

$$X_h(Y(f \circ L_g)) - Y_h(X(f \circ L_g)) = X_h(Y(f) \circ L_g) - Y_h(X(f) \circ L_g) = X_{gh}(Y(f)) - Y_{gh}(X(f)) = [X, Y]_{gh}(f)$$

So $((d_h L_g)[X, Y]_h)(f) = [X, Y]_{gh}(f)$ meaning $[X, Y]$ is left-invariant. ■

So, to obtain a Lie algebra from a Lie group we take the underlying vector space to be $T_1 G$ and define the Lie bracket of $v, w \in T_1 G$ by lifting each v, w to their corresponding left invariant vector fields X^v, X^w and taking $[v, w] = [X^v, X^w]_1$.

After seeing such a construction one might have two questions.

- Is this assignment functorial?
- Is every Lie algebra the Lie algebra of some Lie group?

The answer to both of these questions is yes. However the proof of the second is highly nontrivial and beyond the scope of this text. The first on the other hand is decently simple.

Theorem 3.4. *Let G, H be Lie groups and $f : G \to H$ a Lie group homomorphism. Then the map $d_1 f : T_1 G \to T_1 H$ is a Lie algebra homomorphism.*

Proof. We already know that this is a linear map so we need only show that it preserves the Lie bracket. Now, note that $f \circ L_g = L_{f(g)} \circ f$ and so for any $\gamma : H \to \mathbb{R}$

$$\begin{aligned}(d_0 f \circ d_0 L_g)v)(\gamma) &= (d_0(f \circ L_g)v)(\gamma) \\ &= v(\gamma \circ f \circ L_g) \\ &= v(\gamma \circ L_{f(g)} \circ f) \\ &= (d_0(L_{f(g)} \circ f)v)(\gamma) \\ &= (d_0 L_{f(g)} \circ d_0 f)v)(\gamma)\end{aligned}$$

So for any tangent vector v letting X^v denote the corresponding left-invariant vector field we have $df X^v = X^{d_0 f(v)}$. Therefore

$$\begin{aligned}d_0 f([v, w]) &= df([X^v, X^w])_0 \\ &= df(X^v X^w - X^w X^v)_0 \\ &= (df X^v df X^w - df X^w df X^v)_0 \\ &= (X^{d_0 f(v)} X^{d_0 f(w)} - X^{d_0 f(w)} X^{d_0 f(v)})_0 \\ &= [X^{d_0 f(v)}, X^{d_0 f(w)}]_0 \\ &= [d_0 f(v), d_0 f(w)]\end{aligned}$$

So the derivative is a Lie algebra homomorphism. Further, since the derivative respects identity and composition we therefore obtain functoriality. ■

We denote the functor taking Lie groups to Lie algebras Lie.

4. The exponential map and connected Lie groups

We are now going to discuss the exponential map and how it relates connected Lie groups and their Lie algebras. Note that from here on we will denote the derivative of the left and right multiplication maps by $d_p R_g(v) = vg$ and $d_p L_g(v) = gv$. First we prove a few results about how differentiation and group multiplication interact

Proposition 4.1. *Let $f, g : G \to H$ be Lie group homomorphisms. Then the smooth map fg has derivative $d_p fg = (d_p f)g(p) + f(p)(d_p g)$*

Proof. Let m denote the the multiplication map

$$(d_p fg) = d_{f(p),g(p)} m \circ (d_p f, d_p g)$$

Now let us find $d_{a,b} m$. Then this is a linear map on $T_a H \oplus T_b H$ meaning $d_{a,b} m(v, w) = d_{(a,b)} m(v, 0) + d_{(a,b)} m(0, w)$ Since $(v, 0)(\gamma) = v(\gamma|_a)$ and $(0, w)(\gamma) = w(\gamma|_b)$ we thus have

$$d_{(a,b)} m(v, 0) + d_{(a,b)} m(0, w) = vb + aw$$

Therefore

$$((d_p fg)v)(\gamma) = d_{f(p),g(p)} m \circ (d_p f, d_p g) = (d_p f)g(p) + f(p)(d_p g)$$

■

Now let's talk about the exponential map. This is roughly the map from $T_1 G = \mathrm{Lie}(G) \to G$ obtained by wrapping a tangent vector onto G. To define this we first have to show existence and uniqueness of a certain curve. In order to do so we will be using existence and uniqueness results for ordinary differential equations [1], if you don't know these feel free to black box them.

Proposition 4.2. *For any $v \in \mathrm{Lie}(G)$ there is a unique Lie group homomorphism $\gamma_v : \mathbb{R} \to G$ such that $\gamma'(0) = v$.*

Proof. First note that such a function must have

$$\gamma_v(b)\gamma_v(a) = \gamma_v(b+a) = \gamma_v(a+b) = \gamma_v(a)\gamma_v(b)$$

In particular

$$\gamma_v(2t) = \gamma_v(t)^2$$

so differentiating with respect to a we find

$$\gamma_v'(a+b) = \gamma_v'(a)\gamma_v(b)$$

So if we set $a = 0$ then we have

$$\gamma_v'(b) = x\gamma_v(b)$$

This articulates an ODE with initial condition $\gamma_v(0) = 1$ which by existence and uniqueness for ODEs has a unique solution on some neighborhood $|t| < \delta$. To show that this solution extends to the whole real line we induct on n to show that it exists for $|t| < 2^n\delta$. We already have the base case so now suppose a solution exists on $|t| < 2^{n-1}\delta$, then we define for any $|t| < 2^n\delta$, $\gamma_v(t) = \gamma_v(\frac{t}{2})^2$. We have already seen that this must agree with the previous solution for $|t| < 2^{n-1}\delta$ and since

$$\gamma_v'(t) = \frac{1}{2}(\gamma_v'(\frac{t}{2})\gamma_v(\frac{t}{2}) + \gamma_v(\frac{t}{2})\gamma_v'(\frac{t}{2}) = \frac{1}{2}(x\gamma_v(\frac{t}{2})^2 + x\gamma_v(\frac{t}{2})^2) = x\gamma_v(\frac{t}{2})^2 = x\gamma_v(t)$$

meaning that this is a solution to our differential equation. Thus a solution exists globally meaning our claim holds. ■

The existence of this curve allows us to define the exponential map, which uses the curve to figure out how a vector is "wrapped" onto a Lie group.

Definition 4.3. The exponential map exp : Lie(G) → G is defined by $\exp(v) = \gamma_v(1)$. This immediately tells us that $\exp(tv) = \gamma_v(t)$.

Example. The exponential map of the general linear group (and by extension any matrix Lie group) is given by

$$\exp(X) := \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

where $X \in \mathbb{R}^{n^2}$ is viewed as a real matrix. This follows because

$$\sum_{n=0}^{\infty} \frac{0^n}{n!} = I$$

and because the derivative of $\exp(tX)$ is given by

$$\sum_{n=0}^{\infty} \frac{t^n X^{n+1}}{n!}$$

which when $t = 0$ evaluates to X. Thus $\exp(X) = \gamma_X(1)$.

As with many mathematical objects we only ever care about the useful properties exp has. One such very useful property is as follows

Proposition 4.4. *viewing the tangent space as the smooth manifold* $\mathbb{R}^n$ *we have* $\exp(0) = 1$, $d_0\text{exp} = \text{Id}$ *and that for some for some open subset* U *containing* 0 *such that* $\exp|_U$ *is a diffeomorphism onto its image.*

Proof. Note that $\exp(tv) = \gamma_v(t)$ so $\exp(0) = 1$. Further we have $v = \gamma_v'(0) = \frac{d}{dt}\exp(vt)|_{t=0} = d_0\exp(v)$ So $d_0\exp = \text{Id}$. By the inverse function theorem this immediatlly implies that for some for some open subset U containing 0 such that $\exp|_U$ is a diffeomorphism onto its image. ■

This should give you a nice idea of why exp is in some sense the canonical map from $\text{Lie}(G) \to G$, This also allows us to define an inverse to exp on some open subset of 1, we notate this inverse by log, it is called the logarithm. The exponential map also has a couple more nice properties that allow us to more easily connect $\text{Lie}(G)$ and G.

Proposition 4.5. *We have*

- *if* f *is a Lie group homomorphism then* $f \circ \exp = \exp \circ d_0 f$
- *for any* $v \in \text{Lie}(G)$ *and* $a, b \in \mathbb{R}$ $\exp((a+b)v) = \exp(av)\exp(bv)$

Proof. The functions $(\exp \circ d_0 f)(tv)$ and $(f \circ \exp)(tv)$ both satisfy the differential equation $\gamma'(t) = \gamma(t) d_0 f(v)$ with the same initial conditions, so by uniqueness of solution to ODEs, they are equal.

Now let $a, b \in \mathbb{R}$ then

$$\exp((a+b)v) = \gamma_v(a+b) = \gamma_v(a)\gamma_v(b) = \exp(av)\exp(bv)$$

■

This proposition implies a possible connection between morphisms of Lie groups and their derivatives given by the exponential map. In order to make such a relationship more rigorous we have to discuss connected Lie groups. Connected Lie groups are of particular interest to us because every Lie group is an extension of a discrete group by a connected Lie group. First recall that the connected component of a point p in some topological space is the maximal connected subset containing p. (Note, we will really be using the notion of path connectedness instead of connectedness, this doesn't change anything because the notions coincide for manifolds.)

Proposition 4.6. *Let* G *be a Lie group and let* $G_{(1)}$ *be the connected component of the identity. Then* $G_{(1)}$ *is a normal subgroup and* $G/G_{(1)}$ *with the quotient topology is discrete.*

Proof. First suppose $x, y \in G_{(1)}$ then we have a path f_x from 1 to x and a path f_y from 1 to y. Then xf_y is a path from x to xy meaning via concatanating paths we have a path from 1 to xy. Therefore $xy \in G_{(1)}$. This shows that $G_{(1)}$ is a subgroup, to show it is normal suppose $g \in G$, then gf_xg^{-1} is a path from 1 to gxg^{-1} meaning $gxg^{-1} \in G_{(1)}$. This shows that

$G_{(1)}$ is a normal subgroup. Now note that since multiplication by any $g \in G$ is continuous with continuous inverse it sends open sets to open sets. Since every point in G has an open neighborhood homeomorphic to $\mathbb{R}^n$ it is thus locally connected meaning that its connected components are open [3]. Thus the cosets $gG_{(1)}$ are open and thus $G/G_{(1)}$ is discrete. ■

Connected Lie groups have the wonderful property that neighborhoods of 1 generates the whole group.

Theorem 4.7. *suppose $1 \in U$ is an open subset of some connected Lie group G, then the subgroup generated by U is G.*

Proof. Let H be the subgroup of G generated by U. Then $H = \bigcup_{h \in H} hU$ so H is open. Now suppose $g \in G - H$ then the coset $gH \subset G - H$ contains g and is open, thus $G - H = \bigcup_{g \in G-H} gU$ and thus H is closed. By connectedness the only clopen subset of G is G itself meaning $G = H$. ■

This and theorem along with the exponential map finally allows us to give a very strong relation between Lie group homomorphisms of connected Lie groups and their derivatives.

Theorem 4.8. *Suppose G is a connected Lie group and $f : G \to H$ is a Lie group homomorphism. Then f is determined by its derivative at 1.*

Proof. Suppose $g : G \to H$ is another Lie group homomorphism with the same derivative. Recall that exp is a diffeomorphism onto some open neighborhood U of 1 when restricted to some open neighborhood of 0. Additionally note that $\exp(d_1 f(v)) = f(\exp(v))$, so for any $u \in U$ $g(u) = f(u)$. Now suppose $x \in G$, then since U generates G we have some $u_1 u_2 \ldots u_n = x$ meaning

$$f(x) = f(u_1 u_2 \ldots u_n) = f((u_1) f(u_2) \ldots f(u_n) = g(u_1) g(u_2) \ldots g(u_n) = g(u_1 u_2 \ldots u_n) = g(x)$$

■

This gives us a very strong relationship between the category of Lie groups and Lie algebras. The functor Lie from the category of connected Lie groups to Lie algebras is faithful.

5. The general theory of Lie algebras

In this section we will discuss the general algebraic theory of Lie algebras. We begin by discussing a few basic facts and definitions to do with Lie algebras

Proposition 5.1. *Let $\mathfrak{g}$ be a Lie algebra. Then $[x, y] = -[y, x]$*

Proof.

$$[x, y] + [y, x] = [x + y, y + x] = 0$$

Thus

$$[x, y] = -[y, x]$$

■

We also have the notion of a Lie subalgebra ideal.

Definition 5.2. For any Lie algebra $\mathfrak{g}$ a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a subspace of the underlying vector space such that if $x, y \in \mathfrak{h}$ then $[x, y] \in \mathfrak{h}$. This can be abbreviated as $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. $\mathfrak{h}$ is an ideal if $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

A certain type of Lie algebra is especially interesting.

Definition 5.3. Let A be any associative algebra, then we may define a lie bracket by

$$[x, y] = xy - yx$$

These fulfill bilinearity from the bilinearity of multiplication, the jacobi identity comes from the associativity of multiplication and the alternating properties follows by

$$[x, x] = xx - xx = 0$$

Proposition 5.4. *Let A, B be associative algebras, then if $f : A \to B$ is an algebra homomorphism, it is also a Lie algebra homomorphism*

Proof. If $x, y \in A$ then

$$f([x, y]) = f(xy - yx) = f(x)f(y) = f(y)f(x) = [f(x), f(y)]$$

■

One particularly important such lie algebra is the lie algebra of $n \times n$ matrices denoted $gl_n(\mathbb{R})$. We can of course generalize this to any associative algebra of endomorphisms, including infinite dimensional ones. We denote the Lie algebra of endomorphisms of a given vector space V by $\mathfrak{gl}(V)$. Importantly the endomorphisms of the underlying vector space of a given lie algebra $\mathfrak{g}$ is indeed a lie algebra. One of the key ideas we use to study Lie algebras is that of a representation.

Definition 5.5. A representation of a Lie algebra $\mathfrak{g}$ is a lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$.

In particular every lie algebra has a representation called the adjoint representation.

Definition 5.6. If $\mathfrak{g}$ is a Lie algebra with underlying vector space V then the adjoint representation is the map $x \mapsto \text{ad}_x$ where $\text{ad}_x \in \text{end}(V)$ is defined by

$$\text{ad}_x(y) := [x, y]$$

Linearity follows from linearity of the lie bracket and preservation of the Lie bracket follows because from the alternating property and the Jacobi identity.

In general, having an associative algebra structure is very useful and it would be ideal if every Lie algebra came from an associative algebra. It is fairly trivial to show that there are some Lie algebras that aren't also associative algebras (look at the lie algebra of vector fields for example) but it is actually true that every Lie algebra is a Lie subalgebra of some Associative algebra. In order to show this we will construct a sort of free associative algebra on a Lie algebra called the universal enveloping algebra. In order to do so we will construct a "free" associative algebra on a vector space.

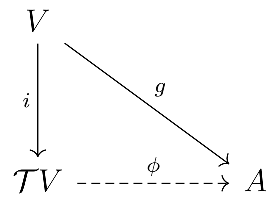
Definition 5.7. For any vector space V the tensor algebra on V, $\mathcal{T}V$ is defined to be

$$\mathcal{T}V := \bigoplus_{n \geq 0} V^{\otimes n}$$

With multiplication defined by $vw \mapsto v \otimes w$. There is a natural injection $i : V \to \mathcal{T}V$ sending $v \mapsto (0, v, 0, \ldots)$.

In particular the tensor algebra fulfills a certain universal property

Proposition 5.8. *Let V be any vector space and A any associative algebra with $g : V \to A$ a linear map. Then there exists a unique associative algebra homomorphism $\phi : \mathcal{T}V \to A$ such that*

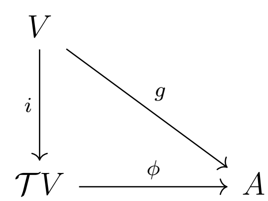


Commutes

Proof. By the universal property of direct sums every map $\mathcal{T}V \to A$ is defined uniquely by maps $V^{\otimes n} \to A$. So define ϕ by

- For any $a \in \mathbb{R}$, $\phi(a) = a1_A$
- For any $v_1 \otimes v_2 \otimes \ldots \otimes v_n$, $\phi(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = g(v_1)g(v_2)\ldots g(v_n)$

Then this map respects multiplication because $\phi(x \otimes y) = \phi(x)\phi(y)$ and is linear because g is linear. Furthermore since $\phi(v) = g(v)$



commutes. As for uniqueness suppose ρ is another such map, then because it respects multiplication

$$\rho(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = \rho(v_1)\rho(v_2)\ldots\rho(v_n) = \rho(i(v_1))\rho(i(v_2))\ldots\rho(i(v_n)) = g(v_1)g(v_2)\ldots g(v_n)$$

and because it respects identity, for any $a \in \mathbb{R}$

$$\rho(a) = \rho(a1) = a\rho(1) = a1_A$$

Thus by the universal property of direct sums $\rho = \phi$ meaning ϕ is unique. ∎

The universal enveloping algebra is now constructed as a quotient of the tensor algebra.

Definition 5.9. For any Lie algebra } with underlying vector space V, define the universal enveloping algebra $U_\mathfrak{g}$ by

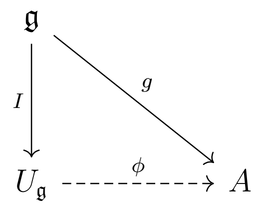
$$U_\mathfrak{g} := \mathcal{T}V/\langle [v, w] - (v \otimes w - w \otimes v)\rangle$$

The natural injection $V \to \mathcal{T}V$ descends to an injective lie group homomorphism $\mathfrak{g} \to U_\mathfrak{g}$ via the surjection $\mathcal{T}V \to U_\mathfrak{g}$. This is a lie group homomorphism by definition of $U_\mathfrak{g}$ and is injective because the ideal of $\mathcal{T}V$ generated by $[x, y] - (x \otimes y - y \otimes x)$ does not contain any elements in the image of i.

Similarly to the tensor algebra $U_\mathfrak{g}$ fulfills a certain nice universal property.

Proposition 5.10. *If $\mathfrak{g}$ is any lie algebra and A is any associative algebra then every lie algebra morphism $g : \mathfrak{g} \to A$ lifts uniquely to an associative algebra morphism $\phi : U_\mathfrak{g} \to A$*

such that



commutes

Proof. Since g is linear it lifts uniquely to a map $\rho : \mathcal{T}V \to A$ which must identify $\rho([x,y]) - \rho(xy - yx)$ and thus descends uniquely to a homomorphism $\phi : U_{\mathfrak{g}} \to A$. ∎

The final theorem of this section will show that every Lie algebra has a faithful (but likely infinite dimensional) representation. That is for every Lie algebra $\mathfrak{g}$ there is some vector space V such that there exists an injective Lie algebra homomorphism $\mathfrak{g} \to \mathrm{end}(V)$.

Theorem 5.11. *Let $\mathfrak{g}$ be a Lie algebra and $U_{\mathfrak{g}}$ its universal enveloping algebra, let V be the underlying vector space of the universal enveloping algebra. Then there exists an injective lie algebra homomorphism $\mathfrak{g} \to \mathrm{end}(V)$.*

Proof. For any $v \in \mathfrak{g}$ define $\phi_v : V \to V$ by $\phi_v(x) = i(v)x$. Then we define a map $\mathfrak{g} \to \mathrm{end}(V)$ by $v \mapsto \phi_v$, this map is a lie algebra homomorphism because i is a lie algebra homomorphism. It is injective because if $v \neq w$, $i(v) \neq i(w)$ and thus $\phi_v(1) = i(v)1 \neq i(w)1 = \phi_w(1)$. ∎

There is a more specific version of this theorem called Ado's theorem which states that every finite dimensional Lie algebra has a finite dimensional representation. This theorem is much harder to prove but much more powerful. It a primary tool used in the proof of Lie's third theorem, that every Lie algebra is the Lie algebra of some Lie group.

6. Conclusion

In this paper, we have developed the tools needed to pass from the global, nonlinear world of Lie groups to the local, linear world of Lie algebras. Starting with the language of smooth manifolds, we introduced tangent spaces and vector fields, and saw how the Lie bracket endows the space of vector fields with a rich algebraic structure.

For a Lie group G, the special class of left-invariant vector fields allowed us to transport this structure to the tangent space T_1Gat the identity. This space, equipped with the induced bracket, is the Lie algebra $\mathrm{Lie}(G)$. We saw that the construction is functorial: a homomorphism of Lie groups induces a homomorphism of their Lie algebras.

The exponential map then provided a canonical way to move back from $\mathrm{Lie}(G)$ to G. In the connected case, neighborhoods of the identity generate the entire group, and Theorem 4.8 showed that a Lie group homomorphism is determined entirely by its derivative at the identity. This gives a strong link between the category of connected Lie groups and the category of Lie algebras: the Lie functor is faithful. Finally, we discussed a few of the more algebraic theory of lie groups. Constructing the universal enveloping algebra and using it to find a faithful representation of any Lie algebra.

The correspondence between Lie groups and lie algebras lies at the heart of Lie theory and underpins many applications, from geometry to physics. The results presented here are only the starting point: further study leads to classification of Lie algebras, representation theory, and the interplay between topology and algebra.

References

[1] Earl A. Coddington and Norman Levinson. *Theory of Ordinary Differential Equations*. McGraw-Hill, 1955.

[2] John M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, 2nd edition, 2012.

[3] James R. Munkres. *Topology*. Prentice Hall, 2nd edition, 2000.