Generalizations of the Isoperimetric Inequality

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Abstract

The classical isoperimetric inequality asserts that among all simple closed curves of fixed length in the plane, the circle encloses maximal area. This article gives a concise proof outline via Steiner symmetrization and then explains how the isoperimetric principle extends beyond the Euclidean plane. On the unit 2-sphere, the minimizers of boundary length for a given area are spherical caps, leading to the sharp inequality $L^2 \geq A(4\pi - A)$. In negatively curved space ($\mathbb{H}^2$), geodesic disks minimize length and the sharp profile becomes $L^2 \geq A(4\pi + A)$. In discrete settings, the appropriate notion of "boundary" is combinatorial: for finite graphs, one studies the edge boundary of a vertex set. We discuss model cases, including the path and cycle, and sketch Harper's edge-isoperimetric theorem on the hypercube via compression. Throughout, the emphasis is on the common theme: curvature and combinatorial structure shape how "volume" is controlled by "boundary."

1 Introduction and brief history

The *isoperimetric problem* asks: among all closed planar curves of given length, which encloses the largest area? Already present in antiquity (often associated with Dido's problem), the modern mathematical treatment matured in the 18–19th centuries. Steiner popularized symmetrization arguments; rigorous existence and regularity issues were settled later via the calculus of variations and geometric measure theory. Today the inequality is a cornerstone of geometry, analysis, and probability, and it admits far-reaching generalizations on manifolds and in discrete structures.

Two viewpoints. There are (at least) two complementary pictures:

- *Rearrangement/symmetrization:* compare an arbitrary set to a symmetric model by equimeasurable rearrangement that does not increase perimeter.
- *Variational/first-variation:* minimize perimeter under a volume constraint; Euler–Lagrange gives constant (geo)desic curvature/mean curvature, which identifies the optimizer.

In the plane these both single out disks. On curved surfaces, curvature modulates the optimal profile but the principles persist. On graphs, rearrangement becomes *compression*, and the variational picture is replaced by local combinatorial improvements.

2 The classical isoperimetric inequality in $\mathbb{R}^2$

We work with sufficiently regular Jordan domains; more generally, the statements extend to measurable sets of finite perimeter with the standard tools of geometric measure theory.

Definition 2.1 (Perimeter and area)**.** For a bounded set $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$, its *area* is

$$A(\Omega) = |\Omega|,$$

where $|\cdot|$ denotes the two-dimensional Lebesgue measure, and its *perimeter* is

$$P(\Omega) = \mathcal{H}^1(\partial\Omega),$$

the one-dimensional Hausdorff measure of the boundary, i.e. the boundary length.

For more general sets of finite perimeter, the perimeter $P(\Omega)$ can be characterized as the total variation of the characteristic function χ_Ω. Here χ_Ω is the function

$$\chi_\Omega(x) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases}$$

and the "total variation" means the distributional gradient $\nabla \chi_\Omega$ is a finite measure, whose mass gives $P(\Omega)$.

Theorem 2.2 (Planar isoperimetric inequality)**.** *For every bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary,*

$$P(\Omega)^2 \;\geq\; 4\pi\, A(\Omega),$$

with equality if and only if Ω is a Euclidean disk.

We will present a robust route via Steiner symmetrization. The method isolates a one-dimensional rearrangement improvement on lines, then integrates it by means of the coarea formula.

Steiner symmetrization and the key inequality

Fix the x-axis. For almost every $x \in \mathbb{R}$ (that is, for all x outside a set of measure zero where the slice may be ill-defined), consider the vertical slice

$$I_x \;=\; \{\, y \in \mathbb{R} : (x, y) \in \Omega \,\}.$$

Each slice I_x is a (possibly disconnected) union of intervals. The *Steiner symmetral* Ω^* (with respect to the x-axis) is obtained by *replacing* each slice I_x with a single centered interval of the same length,

$$I_x^* \;=\; \Big[-\tfrac{|I_x|}{2}, \tfrac{|I_x|}{2} \Big].$$

In words: instead of keeping track of where the mass of Ω sits along the vertical line at x, we redistribute it into one contiguous block, centered on the x-axis, but preserving the one-dimensional measure $|I_x|$.

This construction has two immediate consequences:

(i) **Area preservation:** $A(\Omega^*) = A(\Omega)$, since each slice has the same measure before and after symmetrization.

(ii) **Perimeter monotonicity:** $P(\Omega^*) \leq P(\Omega)$, i.e. symmetrization does not increase boundary length.

Item (ii) is the substantive inequality: rearranging each slice into a single interval reduces the vertical boundary complexity, and the BV slicing machinery upgrades this one-dimensional improvement into a global perimeter inequality.

Lemma 2.3 (Perimeter monotonicity under Steiner)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded set of finite perimeter and let Ω^* be its Steiner symmetral (with respect to the x-axis). Then*

$$P(\Omega^*) \leq P(\Omega),$$

with equality only if almost every vertical slice I_x is a single interval (up to null sets).

Proof sketch. Write $u = \chi_\Omega$. For almost every x, the function $y \mapsto u(x, y)$ is a *BV function* on $\mathbb{R}$, i.e. a function of bounded variation taking values in $\{0, 1\}$. Its one-dimensional total variation along the vertical line $\{x\} \times \mathbb{R}$ equals twice the number N_x of connected components of the slice I_x:

$$\mathrm{Var}_y(u(x, \cdot)) \;=\; 2\, N_x.$$

Replacing $u(x, \cdot)$ by its centered decreasing rearrangement $u^*(x, \cdot)$ preserves the length $|I_x|$ and *minimizes* this 1D variation. Hence,

$$\mathrm{Var}_y(u^*(x, \cdot)) \;\leq\; \mathrm{Var}_y(u(x, \cdot)) \quad \text{for a.e. } x.$$

Vol'pert's slicing theorem for BV functions states that the map

$$x \longmapsto |I_x|$$

is itself a function of bounded variation. Its one-dimensional *total variation* (denoted TV) is given by

$$\mathrm{TV}\big(x \mapsto |I_x|\big) \;=\; |D_x \chi_\Omega|(\mathbb{R}^2),$$

where $\mathrm{TV}(f)$ means the total variation of a real-valued function f on $\mathbb{R}$, and $|D_x\chi_\Omega|$ denotes the total variation measure of the distributional derivative of χ_Ω in the x-direction. Similarly, $|D_y\chi_\Omega|$ measures the variation in the y-direction. Viewing $D\chi_\Omega = (D_x\chi_\Omega, D_y\chi_\Omega)$ as a $\mathbb{R}^2$–valued Radon measure, its total variation is

$$P(\Omega) = |D\chi_\Omega|(\mathbb{R}^2).$$

Steiner symmetrization preserves $|I_x|$ slice-wise, and thus preserves the x-component $|D_x\chi|$. On the other hand, integrating the one-dimensional variation inequality gives

$$|D_y\chi_{\Omega^*}|(\mathbb{R}^2) \;\le\; |D_y\chi_\Omega|(\mathbb{R}^2).$$

Viewing $D\chi$ as a vector-valued Radon measure and using lower semicontinuity together with the convexity of the Euclidean norm, one concludes

$$P(\Omega^*) = |D\chi_{\Omega^*}|(\mathbb{R}^2) \;\le\; |D\chi_\Omega|(\mathbb{R}^2) = P(\Omega).$$

Equality forces $\mathrm{Var}_y(u^*(x,\cdot)) = \mathrm{Var}_y(u(x,\cdot))$ for almost every x, hence $N_x = 1$ almost everywhere. That is, almost every vertical slice is already a single interval. □

Proof of Theorem 2.2 (via iterated Steiner). Apply Steiner symmetrization first with respect to the x-axis to obtain Ω_1, then with respect to the y-axis to obtain Ω_2, and iterate over a dense set of directions. The sequence $\{\Omega_k\}$ has constant area, nonincreasing perimeter, and remains uniformly bounded. Compactness for finite-perimeter sets gives an L^1-limit Ω_∞ with

$$A(\Omega_\infty) = A(\Omega), \qquad P(\Omega_\infty) \le \liminf_k P(\Omega_k) \le P(\Omega).$$

By construction, Ω_∞ is invariant under reflections across a dense set of lines through the origin; continuity of the action yields full rotational invariance, hence Ω_∞ is a disk B_r. Therefore

$$P(\Omega) \ge P(\Omega_\infty) = 2\pi r, \qquad A(\Omega) = A(\Omega_\infty) = \pi r^2,$$

and eliminating r yields $P(\Omega)^2 \ge 4\pi A(\Omega)$. If equality holds, then equality holds at each symmetrization step; by Lemma 2.3 this forces the slices in each symmetrized direction to be single intervals a.e., and the dense family of symmetries forces Ω itself to be a disk up to null sets. □

Remark 2.4 (Strengthenings and stability)**.** For *convex* planar sets, Bonnesen's inequality refines the basic isoperimetric deficit:

$$P(\Omega)^2 - 4\pi A(\Omega) \;\ge\; \pi^2\,(R-r)^2,$$

where r and R denote the inradius and circumradius of Ω.

Beyond exact inequalities, there are also *stability results*: if the deficit

$$\delta := P(\Omega)^2 - 4\pi A(\Omega)$$

is small, then Ω must be quantitatively close to a disk. One convenient way to measure "closeness" is the *Fraenkel asymmetry*, defined by

$$\mathcal{A}(\Omega) \;=\; \inf_{x,r} \frac{|\Omega \triangle B_r(x)|}{A(\Omega)},$$

where $\triangle$ denotes the symmetric difference and the infimum is over all disks $B_r(x)$ of area $A(\Omega)$. In words: $\mathcal{A}(\Omega)$ measures the relative fraction of area where Ω fails to overlap with the best-fitting disk.

Quantitative stability then says

$$\mathcal{A}(\Omega) \;\lesssim\; \frac{\sqrt{\delta}}{P(\Omega)},$$

where the notation $X \lesssim Y$ means that X is bounded above by a constant multiple of Y (with the constant universal, not depending on Ω).

A variational aside: constant curvature

If one minimizes $P(\Omega)$ subject to $A(\Omega) = A_0$, the first variation gives

$$\kappa \equiv \lambda \quad \text{on } \partial\Omega,$$

i.e. the boundary has constant curvature κ, hence is a circle. This complements the rearrangement route and generalizes to higher dimensions (constant mean curvature).

3 Isoperimetry in spaces of constant curvature

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere with the induced metric; let $\mathbb{H}^2$ denote the hyperbolic plane of curvature -1.

Definition 3.1 (Spherical caps)**.** Fix $p \in S^2$ and $\theta \in (0, \pi)$. The *cap* of angular radius θ is

$$C_\theta(p) \;=\; \{\, x \in S^2 : \arccos\langle x, p\rangle \le \theta \,\}.$$

Its boundary is a geodesic circle of geodesic curvature $\cot\theta$.

Theorem 3.2 (Spherical isoperimetric inequality)**.** *For any regular region $\Omega \subset S^2$ with area $A \in (0, 4\pi)$ and boundary length L,*

$$L^2 \;\ge\; A\,(4\pi - A),$$

with equality if and only if Ω is a spherical cap (up to measure zero).

Proof sketch (two routes). (Symmetrization). Perform *zonal symmetrization* about a pole: along each circle of latitude, replace the arc(s) $\Omega \cap \{\text{latitude} = \ell\}$ by the centered arc of the same length. As in Lemma 2.3, one-dimensional rearrangement along latitudes decreases boundary count; integrating in the orthogonal direction preserves area. The result is a rotationally symmetric set C with $A(C) = A(\Omega)$ and $L(\partial C) \le L(\partial\Omega)$; symmetry forces C to be a cap.

(Variational). Existence of perimeter minimizers under an area constraint follows from compactness on S^2. First variation yields boundary with constant geodesic curvature; simple closed curves of constant geodesic curvature on S^2 are geodesic circles, bounding caps. For a cap of angular radius θ,

$$A = 2\pi(1 - \cos\theta), \qquad L = 2\pi\sin\theta.$$

Eliminate θ via $\cos\theta = 1 - \frac{A}{2\pi}$ to obtain $L^2 = A(4\pi - A)$. □

Theorem 3.3 (Hyperbolic isoperimetric inequality)**.** *In the hyperbolic plane $\mathbb{H}^2$ of curvature -1, for any regular region of area A and boundary length L,*

$$L^2 \;\ge\; A\,(4\pi + A),$$

with equality precisely for geodesic disks.

Proof sketch. Geodesic disks minimize boundary for fixed area by either a symmetrization argument in a disk model or by first variation (constant geodesic curvature implies geodesic circles). If D_r is a disk of radius r, then $L = 2\pi\sinh r$ and $A = 2\pi(\cosh r - 1)$. Hence

$$L^2 \;=\; 4\pi^2\sinh^2 r \;=\; 4\pi^2(\cosh^2 r - 1) \;=\; 4\pi^2\big(\big(1 + \tfrac{A}{2\pi}\big)^2 - 1\big) \;=\; 4\pi A + A^2 \;=\; A(4\pi + A).$$

□

Remark 3.4 (Curvature comparison)**.** Compare the three constant-curvature profiles (for $K \in \{-1, 0, 1\}$):

$$L^2 \;\ge\; A\big(4\pi - K\,A\big).$$

Positive curvature ($K = +1$) makes enclosing large regions cheaper; negative curvature ($K = -1$) makes it more expensive. All three are symmetric under complement when $K = +1$, consistent with $A \mapsto 4\pi - A$ on S^2.

4 Higher-dimensional Euclidean isoperimetry

Write $\omega_n = |B_1(0)|$ for the volume of the unit ball in $\mathbb{R}^n$.

Theorem 4.1 (Euclidean isoperimetric inequality in $\mathbb{R}^n$)**.** *For a finite-perimeter set $\Omega \subset \mathbb{R}^n$,*

$$P(\Omega) \ \geq \ n\,\omega_n^{1/n}\,|\Omega|^{\frac{n-1}{n}}, \qquad \textit{equivalently} \qquad P(\Omega)^n \ \geq \ n^n\,\omega_n\,|\Omega|^{n-1},$$

with equality if and only if Ω is a ball (a.e.).

Several proofs exist: Brunn–Minkowski (via concavity of $t \mapsto |(1-t)\Omega + tB|^{1/n}$), calibration with the Newtonian potential, and the Pólya–Szegő inequality (decreasing symmetric rearrangement does not increase the Dirichlet energy, leading to a sharp isoperimetric inequality through the coarea formula).

Remark 4.2 (First variation)**.** Minimizers under a fixed volume constraint necessarily have boundary with *constant mean curvature* (CMC). Alexandrov's theorem shows that embedded closed hypersurfaces in $\mathbb{R}^n$ with constant mean curvature which are topological spheres must in fact be round spheres, recovering the equality cases in Theorem 4.1.

5 Discrete isoperimetry on graphs

Let $G = (V, E)$ be a finite simple graph. For $S \subset V$, the *edge boundary* is

$$\partial S \ = \ \{\, \{u, v\} \in E : u \in S,\ v \in V \setminus S \,\}, \qquad b(S) := |\partial S|.$$

One seeks lower bounds on $b(S)$ in terms of $|S|$ and the structure of G.

Two warm-up examples

Paths and cycles. If $G = P_n$ is the path on n vertices, the sets minimizing $b(S)$ for fixed $|S| = k$ are contiguous intervals; for $1 \leq k \leq n-1$ one gets $b(S) = 1$ if S touches exactly one endpoint and $b(S) = 2$ otherwise. On the cycle C_n, every nonempty proper S satisfies $b(S) \geq 2$, with equality for contiguous arcs. These mirror the 1D Euclidean picture: intervals minimize boundary for fixed measure.

The hypercube and compression

Let Q_n denote the n-dimensional hypercube with vertex set $\{0,1\}^n$ and edges between Hamming-neighbors. For $S \subset \{0,1\}^n$, $b(S)$ counts edges leaving S.

Theorem 5.1 (Edge-isoperimetric theorem on the hypercube (informal form))**.** *Among all $S \subset \{0,1\}^n$ with $|S| = m$, the sets that minimize the* edge *boundary $b(S)$ are* lexicographic initial segments *(in particular, for $m = 2^k$ the minimizers are k-dimensional subcubes). For the* vertex *boundary, the minimizers are* Hamming balls *(initial segments by Hamming weight).*

Proof sketch via compression. For $i \in \{1, \ldots, n\}$, partition $\{0,1\}^n$ into pairs differing only in the i-th coordinate. The *i-compression* C_i replaces S by a set $C_i(S)$ where, within each pair, mass is moved to the member with i-th coordinate 0 whenever the pair was split. Then $|C_i(S)| = |S|$, and one checks locally that $b(C_i(S)) \leq b(S)$: edges in directions $j \neq i$ are unaffected and edges in direction i can only be removed. Iterating $C_1, \ldots, C_n$ in any order yields a *down-set*; among down-sets of a given size, lexicographic initial segments minimize $b(S)$, giving the claim. (For vertex boundary, a related compression and layer-counting argument leads to Hamming balls.) □

A convenient lower bound that captures the correct growth in m is:

Proposition 5.2 (A clean lower bound)**.** *For every $S \subset \{0,1\}^n$ with $m = |S|$,*

$$b(S) \ \geq \ m\,\log_2\!\left(\frac{2^n}{m}\right).$$

Proof idea. It suffices to check lexicographic initial segments; for $m = 2^k$ these are k-dimensional subcubes with $b = m(n-k) = m\log_2(2^n/m)$, and convexity/monotonicity in m gives the general bound (the discrete analogue of $P \gtrsim |S|^{(n-1)/n}$). ☐

Remark 5.3 (Expansion and Cheeger viewpoint)**.** For a d-regular graph G, the *edge-isoperimetric number* (Cheeger constant) is

$$h(G) \;=\; \min_{\emptyset \neq S \subset V,\ |S| \leq |V|/2} \frac{b(S)}{|S|}.$$

Cheeger's inequality relates $h(G)$ to the spectral gap of the Laplacian, but on product graphs like Q_n the purely combinatorial compression method identifies sharp extremals directly. In expander graphs, $h(G)$ is bounded away from 0, mirroring linear isoperimetric growth.

6 A probabilistic cousin: Gaussian isoperimetry (briefly)

In $(\mathbb{R}^n, \gamma_n)$ with standard Gaussian measure γ_n, the correct boundary notion is the *Gaussian surface measure*. The sharp inequality states that among all sets of a given Gaussian measure, *half-spaces* minimize Gaussian surface measure. Equivalently, if $\mu = \gamma_n(A)$ and Φ is the standard normal CDF,

$$\text{Gauss-perimeter}(A) \;\geq\; \phi\big(\Phi^{-1}(\mu)\big),$$

where ϕ is the standard normal density. This isoperimetry controls concentration of measure and underlies many inequalities in high-dimensional probability.

7 Comparisons and Unifying Themes

Across these settings, isoperimetry expresses a universal principle: *boundary controls volume.* Yet geometry and structure shape the sharp form:

- In $\mathbb{R}^2$, Steiner symmetrization drives sets to disks, yielding $P^2 \geq 4\pi A$ and stability of near-equality.
- On S^2 and $\mathbb{H}^2$, constant curvature bends the profile to $L^2 \geq A(4\pi - A)$ and $L^2 \geq A(4\pi + A)$, with geodesic circles extremal.
- In $\mathbb{R}^n$, balls uniquely optimize $P \geq n\,\omega_n^{1/n}|\Omega|^{(n-1)/n}$; first variation gives constant mean curvature and Alexandrov rigidity.
- On graphs, product structure makes compression effective, identifying lexicographic initial segments as edge-isoperimetric extremals in Q_n (subcubes at powers of two) and giving entropy-flavored lower bounds like $b(S) \geq m\log_2(2^n/m)$; for vertex boundary, Hamming balls are extremal.

Curvature (continuous) and product/combinatorial structure (discrete) thus play analogous roles: they constrain optimal shapes and determine the isoperimetric profile.

Technical note on finite-perimeter sets

For completeness: a measurable set $\Omega \subset \mathbb{R}^n$ has *finite perimeter* if $\chi_\Omega \in BV(\mathbb{R}^n)$; its perimeter is $P(\Omega) = \|D\chi_\Omega\|(\mathbb{R}^n)$. The coarea formula for $u \in BV$,

$$\int_{\mathbb{R}} \mathcal{H}^{n-1}(\partial^*\{u > t\})\,dt \;=\; \|Du\|(\mathbb{R}^n),$$

underlies the passage from 1D slice inequalities to global isoperimetry. Lower semicontinuity of total variation ensures compactness under L^1 convergence in symmetrization schemes.

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