Plateau's Problem

Jason Tae

8/14/2025

1 Motivation: Minimal Surfaces in Nature

When a metal wire is dipped in a soap solution, **surface tension** forces the resulting film to be of minimal surface area; a soap film spanning a wire loop minimizes surface area subject to that boundary.

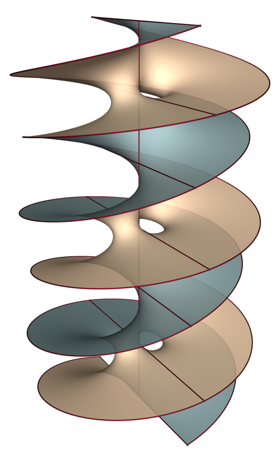
Plateau's Problem, first raised by Joseph-Louis Lagrange in 1760, claims that there exists such a minimal surface for every simple wire loop; soap films are an example of this.

We can now extend this to a mathematical context. The metal wire becomes our **Jordan curve** Γ, while our soap film becomes the desired minimal surface.

Note that all parts of the problem covered in this paper will be solely within $\mathbb{R}^3$.

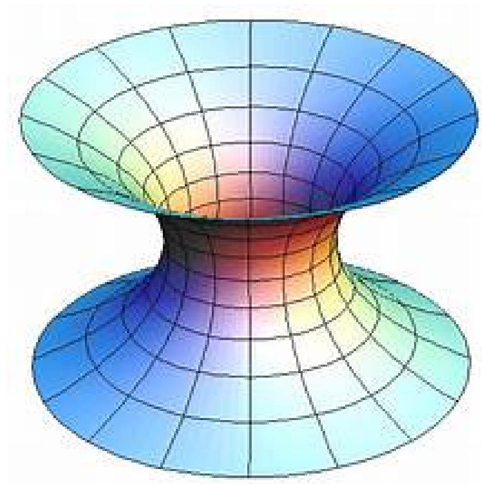
Some examples of minimal surfaces within $\mathbb{R}^3$:

- **Helicoid**: The minimal surface formed along a helix:



A helicoid is parametrized as $\sigma(u, v) = (v \cos u, v \sin u, u)$.

- **Catenoid**: The minimal surface formed along a catenary:



A catenoid is parametrized as $\sigma(u, v) = (\cosh v \cos u, \cosh v \sin u, v)$.

There are many other examples of minimal surfaces, but these are the most common.

2 Plateau's Problem: Statement

For some Jordan curve $\Gamma \in \mathbb{R}^3$, some surface S exists, such that:

1. S **spans** the Jordan curve: $\partial S = \Gamma$,
2. S **minimizes** its area among all surfaces spanning Γ: $A(S) \leq A(S')$ for any other surface S'.

Note: A (classical) minimal surface is one with mean curvature $H \equiv 0$ in the interior. Area-minimizing implies minimality, but not conversely.

For the sake of simplicity, we will not discuss self-intersecting curves.

3 Plateau's Problem: Douglas's Solution

Let D be the **closed unit disk**:

$$D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}.$$

Fix a **continuous, one-to-one, counterclockwise** parametrization:

$$\gamma : S^1 \to \Gamma.$$

Our goal is to find a surface of minimal area that spans Γ.

When we try to minimize directly over maps $u : D \to \mathbb{R}^3$ with $u|_{\partial D} = \gamma \circ \phi$, the boundary reparametrization $\phi : S^1 \to S^1$ creates a **huge symmetry**: composing with disk automorphisms leads to a lack of compactness.

Douglas's fix: restricting to **orientation-preserving homeomorphisms** ϕ that fix 3 boundary points (e.g. 1, $e^{2i\pi/3}$, $e^{4i\pi/3}$). This 3-point normalization helps us further work towards the existence of a minimizer.

Given ϕ, we define our boundary map:

$$f_\phi := \gamma \circ \phi : S^1 \to \mathbb{R}^3.$$

Let $u_\phi : D \to \mathbb{R}^3$ be the unique **harmonic extension** (Poisson integral) of f_ϕ.

Note that among all maps with the same boundary data f_ϕ, u_ϕ minimizes the **Dirichlet energy:**

$$E[u] := \frac{1}{2}\int_D |\nabla u|^2 dxdy = \frac{1}{2}\int_D (|u_x|^2 + |u_y|^2) dxdy.$$

Now, note that given **conformal** u, area is equivalent to energy:

$$A[u] = E[u],$$

as:

$$A[u] = \int_D |\partial_x u \times \partial_y u| = \frac{1}{2}\int_D |\nabla u|^2 \text{ when } u \text{ is conformal.}$$

Thus, if we can confirm that our energy-minimizing harmonic extension u is indeed conformal, then we will have our area minimizer.

The Douglas Functional: Douglas discovered a boundary-only functional $D(\phi) = E[u_\phi]$, which measures a boundary roughness of f_ϕ. Thanks to our equality, we note that minimizing $D(\phi)$ over ϕ is equivalent to minimizing the interior energy among harmonic fillings of reparametrized boundaries.

For reference, a form of Douglas's functional is the integral:

$$D(\phi) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{|\gamma(\phi(e^{it})) - \gamma(\phi(e^{is}))|^2}{2(1-\cos(t-s))} dtds.$$

The derivation of the formula is too complex for this paper, but a short derivation using Fourier modes of the Poisson extension shows $E[u_\phi] = D(\phi)$.

Douglas also showed that $D(\phi)$ is **bounded below** and **lower semicontinuous** on the normalized class of ϕ's, as well as the fact that any minimizing sequence ϕ_k has a **convergent subsequence**.

Thus, there exists a minimizer ϕ_*, such that:

$$D(\phi_*) = \inf_{\phi} D(\phi).$$

Now, let $u := u_{\phi_*}$ be the harmonic map with boundary f_{ϕ_*}.

Observe the first variation of D under smooth one-parameter perturbations ϕ_t, with $\phi_0 = \phi_*$. Stationarity of D at ϕ_* translates, after pulling through the Poisson extension, into **boundary conditions** for the harmonic map u:

$$\langle u_r, u_\theta \rangle|_{\partial D} = 0 \textbf{ and } |u_r| = |u_\theta| \textbf{ on } \partial D,$$

where r, θ are the polar coordinates on our disk. These conditions state that the boundary principal directions are **orthogonal**, and that the boundary metric coefficients are **equal**, respectively.

Thus, our energy minimizer u is both harmonic and conformal.

Finally, we must relate conformality to **area minimality**. For any competitor $v : D \to \mathbb{R}^3$ with the same boundary trace as u:

$$A[v] \geq E[v] \geq E[u] = A[u],$$

where:

- $A[v] \geq E[v]$, and $A[v] = E[v] \iff v$ is conformal,
- $E[v] \geq E[u]$ because u is the harmonic energy minimizer for the boundary data,
- $A[u] = E[u]$ because u is conformal.

Therefore, u has the least area among all spanning maps with the given boundary trace. This proves existence of a solution to Plateau's problem for Γ $\square$

References

[1] J. Douglas, *Solution of the problem of Plateau*, *Trans. Amer. Math. Soc.* **33** (1931), 263–321.

[2] R. Courant, *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces*, Interscience Publishers, New York, 1950.

[3] O. Chodosh, *Some new generic regularity results for minimal surfaces and mean curvature flows*, Geometric Analysis Festival (2021), lecture notes. Available at https://web.stanford.edu/~ochodosh/GAFestival2021.pdf (accessed Aug. 14, 2025).

[4] *Plateau problem*, *Encyclopedia of Mathematics.* Available at https://encyclopediaofmath.org/wiki/Plateau_problem (accessed Aug. 14, 2025).

[5] *Plateau's problem, Wikipedia, The Free Encyclopedia*. Available at https://en.wikipedia.org/wiki/Plateau%27s_problem (accessed Aug. 14, 2025).

[6] R. Osserman, *A Survey of Minimal Surfaces*, Dover, Mineola, 1986. Available at https://archive.org/details/surveyofminimals0000osse (accessed Aug. 14, 2025).

[7] U. Dierkes, S. Hildebrandt, F. Sauvigny, *Minimal Surfaces*, Vol. 1, 2nd ed., Springer, 2010. See Ch. 6.4 on the Douglas functional. Available at https://link.springer.com/book/10.1007/978-3-642-11698-8 (accessed Aug. 14, 2025).

[8] V. Guillemin, B. Kostant, S. Sternberg, *Douglas' solution of the Plateau problem*, *Proc. Natl. Acad. Sci. USA* **85** (1988), 3277–3278. Open access at https://pmc.ncbi.nlm.nih.gov/articles/PMC280191/ (accessed Aug. 14, 2025).