Morse Theory

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Abstract

In Morse theory, we study the topology of smooth manifolds by analyzing smooth functions and the flow lines that connect their critical points. In this paper, we will state and use the Morse inequalities, work through examples (spheres, tori, complex projective spaces, and closed surfaces of genus g). We skim through the overarching principles of Morse theory to gain a general understanding of what it is about.

1 Notation

Let M be a smooth compact n-dimensional manifold without a boundary (unless we say otherwise). We say that a smooth function $f : M \to \mathbb{R}$ has a differential df and, at a critical point, a Hessian $d^2 f$. We use a fixed Riemann metric on M to identify covectors with vectors and also to write the gradient ∇f. When we speak about "the flow," we mean the negative gradient flow unless otherwise specified:

$$\dot{x} = -\nabla f(x(t)).$$

We call a point $p \in M$ critical if $(df)_p = 0$. If the Hessian $d^2 f(p)$ is nondegenerate, then p is nondegenerate and has an integer index $\text{ind}(p) \in 0, \dots, n$ that is equal to the number of negative eigenvalues of $d^2 f(p)$. A smooth function is Morse if all of its critical points are nondegenerate. We write
$Crit_k(f)$ for the set of index-k critical points.

Given a complete vector field X on M with flow φ_t, the stable and unstable manifolds of a rest point p are

$$W^s(p) = \{x \in M : \lim_{t \to +\infty} \phi_t(x) = p\}, W^u(p) = \{x \in M : \lim_{t \to -\infty} \phi_t(x) = p\}.$$

We will use $\mathcal{M}(a, b)$ to denote the set of flow lines $\gamma : \mathbb{R} \to M$ solving $\dot{\gamma} = -X(\gamma)$ with $\lim_{t \to -\infty} \gamma(t) = a$ and $\lim_{t \to +\infty} \gamma(t) = b$. Using the quotient by time-translation gives us

$$\widehat{\mathcal{M}}(a, b) = \mathcal{M}(a, b)/\mathbb{R}.$$

2 Morse Theory: General

Morse theory is based on a precise normal form for f near a nondegenerate critical point p.

Morse Lemma: There are local coordinates $(x_1, \ldots, x_n)$ centered at p in which

$$f(x) = f(p) - x_1^2 - x ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_n^2,$$

where $\lambda = \text{ind}(p)$. This formula tells us that nondegenerate critical points are isolated; the stable manifold $W^s(p)$ is an embedded open $(n-\lambda)$-disk; the unstable manifold $W^u(p)$ is an embedded open λ-disk; and that the flow decreases f strictly away from critical points.

We can visualize this with these examples: on the sphere S^2 with the height function, there is a minimum of index 0 and a maximum of index 2, and there are no saddles. On the torus T^2, a generic height function has four critical points: the minimum (0), two saddles $(1, 1)$, and the maximum (2).

Although we can work with the metric gradient $-\nabla f$, it is more convenient to allow a larger class of vector fields adapted to f.

Pseudo-gradients. A vector field X is a pseudo-gradient for f if $df(X) \leq 0$ with equality only at critical points. Near each critical point p, there are Morse coordinates where X and the negative Euclidean gradient of the quadratic normal form coincide. This means that the local dynamics of X align with those of $-\nabla f$, while providing global flexibility (useful for achieving transversality).

Morse–Smale condition. For critical points a, b, the stable and unstable manifolds $W^s(b)$ and $W^u(a)$ are immersed sub manifolds. The pair (f, X) is Morse–Smale if all these manifolds intersect transversally, written as $W^u(a) \pitchfork W^s(b)$. Under this, we can write the intersection $\mathcal{M}(a, b) = W^u(a) \cap W^s(b)$ as a smooth manifold of dimension $\text{ind}(a) - \text{ind}(b)$, and its time-translation quotient $\widehat{\mathcal{M}}(a, b)$ has one less dimension.

Transversality is crucial because it makes trajectories' spaces behave like manifolds; in particular, zero-dimensional moduli spaces are finite sets that can be counted, one-dimensional ones are unions of intervals, and so on.

3 Compactness and Broken Trajectories

An important part of $\widehat{\mathcal{M}}(a, b)$ is that it has a natural compactification by adding broken trajectories. Suppose $\text{ind}(a) - \text{ind}(b) = 2$. Then $\widehat{\mathcal{M}}(a, b)$ is a one-dimensional manifold. Flow lines cannot "wander off" except by breaking near intermediate critical points, which is a sequence of unparametrized trajectories that can limit a concatenation to $a \to c \to b$ where $\text{ind}(c) = \text{ind}(a) - 1$.

The compact nature that we get by conjoining these endpoints turns $\widehat{\mathcal{M}}(a, b)$ into a compact 1-manifold with boundary. The boundary points correspond precisely to these broken trajectories.

4 The Morse Complex

Fix a Morse function f and a Morse–Smale pseudo-gradient X. For a coefficient ring (or field) $\Bbbk$, the chain group in degree k is the free $\Bbbk$-module

$$C_k(f; \daleth) = \bigoplus_{a \in Crit_k(f)} \daleth \cdot a.$$

The boundary operator counts index-drop-one trajectories. Let $a \in Crit_k(f)$:

$$\partial a = \sum_{b \in Crit_{k-1}(f)} n_X(a,b) b,$$

where $n_X(a,b)$ is the number of points in the zero-dimensional space $\widehat{\mathcal{M}}(a,b)$. If $\Bbbk = \mathbb{Z}/2$, then this is the parity of that finite set.

$\partial^2 = 0$**.** When $\mathrm{ind}(a) - \mathrm{ind}(b) = 2$, the compact one-dimensional manifold $\overline{\widehat{\mathcal{M}}(a,b)}$ has a finite boundary consisting of broken trajectories $a \to c \to b$ with $\mathrm{ind}(c) = k-1$. Each broken trajectory contributes exactly one boundary point, and the boundary of a compact 1-manifold has an even number of points. We can read the coefficient of b in $\partial(\partial a)$ as the boundary count $\partial^2 = 0$.

The resulting homology groups are $H_k(C_*, \partial)$, and these are the Morse homology of M with coefficients in $\Bbbk$. We will soon identify them with the singular homology of M.

4.1 Orientations and Integer Coefficients

To define ∂ over $\mathbb{Z}$, we have to choose orientations of stable manifolds $W^s(p)$ for all critical points p. This choice orients the unstable manifolds and orients transverse intersections $W^u(a) \cap W^s(b)$. This means that $\widehat{\mathcal{M}}(a,b)$ is an oriented zero-dimensional manifold, and we let $N_X(a,b) \in \mathbb{Z}$ be the algebraic count of its points. The signed boundary map is then

$$\partial a = \sum_{b \in Crit_{k-1}(f)} N_X(a,b), b$$

which still satisfies $\partial^2 = 0$ by the same compactness-by-breaking argument.

4.2 Theorems and Exploration

We will summarize the main theorems that follow from everything we covered above

Theorem 5.2.1 (Morse Lemma). Near a nondegenerate critical point p of index λ, there are coordinates in which f is exactly $f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$.

Theorem 5.2.2 (Stable/unstable disks). For an adapted pseudo-gradient X, $W^u(p)$ and $W^s(p)$ are embedded open disks of dimensions $\mathrm{ind}(p)$ and $n - \mathrm{ind}(p)$.

Theorem 5.2.3 (Transversality and dimensions). If (f, X) is Morse–Smale, then $\mathcal{M}(a,b) = W^u(a) \cap W^s(b)$ is a smooth manifold of dimension $\mathrm{ind}(a) - \mathrm{ind}(b)$, and $\widehat{\mathcal{M}}(a,b)$ has one dimension less.

Theorem 5.2.4 (Compactness by breaking). If $\mathrm{ind}(a) - \mathrm{ind}(b) = 2$, then $\overline{\widehat{\mathcal{M}}(a,b)}$ is a compact 1-manifold whose boundary consists of broken trajectories $a \to c \to b$ with $\mathrm{ind}(c) = \mathrm{ind}(a) - 1$.

Corollary 5.2.5 ($\partial^2 = 0$). The boundary operator is defined by counting index-drop-one trajectories satisfies $\partial^2 = 0$.

Theorem 5.2.6 (Invariance). The homology of (C_*, ∂) does not depend on the choice of f or X and agrees with the singular homology of M.

5 Cellular Picture and Invariance

The unstable manifolds of a Morse–Smale pair (f, X) give a decomposition of M with precisely one k-cell for each $a \in$
$Crit_k(f)$. We can read the attaching maps from the flow near the intersections $W^u(a) \cap W^s(b)$, and the cellular boundary operator coincides with the Morse boundary. Collapsing the flow lines tells us that each unstable manifold with an open disk whose boundary attaches along lower-index unstable manifolds.

A different route to invariance is by using continuation: given two Morse–Smale pairs (f_0, X_0) and (f_1, X_1), we know that one of them constructs a chain map by counting flow lines in $M \times \mathbb{R}$ for a time-dependent interpolation.

6 Morse Inequalities and Euler Characteristic

Let $c_k =$
$Crit_k(f)$ and let b_k be the k-th Betti number of M with coefficients in $\Bbbk$. The weak Morse inequalities say that $c_k \geq b_k$ for every k. The strong Morse inequalities compare the alternating partial sums:

$$\sum_{i=l}^{m} (-1)^{m-i} c_i \geq \sum_{i=l}^{m} (-1)^{m-i} b_i (0 \leq l \leq m \leq n)$$

Taking $\ell = 0$ and $m = n$ yields equality

$$\sum_{k=0}^{n} (-1)^k c_k = \sum_{k=0}^{n} (-1)^k b_k = \chi(M)$$

so any Morse function has some critical points whose total alternating sum equals the Euler characteristic. When equalities hold term by term, the function is called perfect.

7 Worked Examples

We will look at various cases of the Morse complex.

7.1 The n-sphere

On $S^n \subset \mathbb{R}^{n+1}$, the height function has exactly two critical points: a minimum of index 0 and a maximum of index n. By choosing X so that there are no additional connecting orbits except the ones forced by index considerations, we get $C_0 \cong C_n \cong \Bbbk$ and $C_k = 0$. The boundary operator is zero and $HM_0 \cong HM_n \cong \Bbbk$, $HM_k = 0$ for $0 < k < n$, as expected.

7.2 The two-torus

On $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, consider $f(x, y) = \cos(2\pi x) + \cos(2\pi y)$. There are four critical points with indices $(0, 1, 1, 2)$. For a generic pseudo-gradient, there are two index-drop-one trajectories out of each saddle, one to the minimum and one to the maximum, which we can arrange so that the signed counts cancel in ∂ for degree 1. The complex therefore has $C_0 \cong C_2 \cong \Bbbk$, $C_1 \cong \Bbbk^2$, with $\partial = 0$ and $HM_1 \cong \Bbbk^2$.

7.3 Complex projective space

On $\mathbb{C}P^n$, a generic (real) moment-map-like function has one critical point in each even index $0, 2, \ldots, 2n$ and none in odd degrees. The Morse complex has one generator in each even degree and $HM_{2k} \cong \Bbbk$ for $k = 0, \ldots, n$, $HM_{2k+1} = 0$, matching the standard cohomology ring structure (after dualizing).

7.4 Closed oriented surfaces

Let Σ_g be a closed surface of genus $g \geq 1$. Any Morse function must have at least 2 critical points in degrees 0 and 2 and at least $2g$ saddles (by the Euler characteristic $\chi(\Sigma_g) = 2 - 2g$). We can choose f to achieve one minimum, one maximum, and $2g$ saddles. The Morse complex then has $C_0 \cong C_2 \cong \Bbbk$, $C_1 \cong \Bbbk^{2g}$. The index forces $\partial = 0$ on C_2 and C_1, and $HM_1 \cong \Bbbk^{2g}$, which recovers $H_1(\Sigma_g)$.

8 Relative Morse Theory and Boundary

If M has a boundary, we decompose ∂M into an incoming part $\partial^- M$ and an outgoing part $\partial^+ M$, which is determined by the sign of $\langle X, \nu \rangle$ where ν is the outward normal. Assuming that f has no critical points on ∂M and X is everywhere transverse to ∂M, one obtains a relative Morse complex whose homology computes $H_*(M, \partial^+ M)$. The associated long exact sequence for the pair is compatible with inclusions and can be read off from filtrations by sublevel sets of f.

9 Functoriality, Products, and Duality

Morse homology is functorial under smooth maps respecting the gradient flow. For example, at the chain level, we can construct maps by counting solutions of hybrid flow equations associated with the map. On product manifolds, we get a Künneth formula

$$HM_*(M \times N) \cong HM_*(M) \otimes HM_*(N)$$

for suitable coefficients. On closed oriented n-manifolds, reversing the function $(-f)$ interchanges stable and unstable directions and gives us a dual complex, leading to Poincaré duality $HM_k(M) \cong HM^{n-k}(M)$.

References

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