THE CALCULUS OF VARIATIONS

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Abstract. In this paper, we will introduce calculus of variations, making use of the Euler-Lagrange equation to discover the cycloid as the brachistochrone. We will then briefly discuss minimal surfaces as another application of both calculus of variations itself and a generalization of the Euler-Lagrange equation.

1. Introduction

The spirit of calculus of variations was first publicly applied to solve the puzzle of the brachistochrone. After its use, inspired by Fermat's earlier idea that nature optimizes for minimum time in optics, Maupertuis was the first to apply this principle to mechanics.

He optimized for a function he called the action, and notable developments in numerous fields of physics would later come from examining the action using calculus of variations. Euler had earlier derived a useful equation for working in the calculus of variations, but Lagrange did it with far less tedium, both doing it to develop action principles. We know it as the Euler-Lagrange equation:

$$\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0$$

We will first discuss the specific type of function that calculus of variations works with, called a functional; the action is one such function. Using the fundamental lemma of calculus of variations, we will then derive the Euler-Lagrange equation.

To be precise, the above is the one-dimensional equation; there are multiple generalizations of this equation if a problem needs it. We will mainly focus on the brachistochrone problem. One of the generalizations of the Euler-Lagrange equation briefly comes after as an approach to minimal surfaces, which are surfaces with mean curvature $H = 0$ — they locally minimize area, and are commonly realized as the behavior of soap films.

2. Functionals

Let $\mathbf{X}$ be some space, which can be thought of as a set with additional structure. We define a functional more generally in analysis as a mapping $\mathbf{X} \to \mathbb{R}$ or $\mathbf{X} \to \mathbb{C}$, but for the purposes of calculus of variations, $\mathbf{X}$ will be a function space specifically — a set of functions from some domain to some codomain — and we will work in $\mathbb{R}$. So, a functional in calculus of variations can be thought of as a real-valued function of real-valued functions. We will assume from this point that the input functions are sufficiently nice.

Definition 1. A *functional* is a function $S : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}$, where $\mathbb{R}^{\mathbb{R}}$ is the set of functions $f : \mathbb{R} \to \mathbb{R}$.

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To illustrate notation, we can use arclength as an example of a functional. Let γ be some curve, and let $t_1 \leq t_2$ be in the domain of γ; then the arclength from t_1 to t_2 is the functional

$$\gamma \mapsto S[\gamma] = \int_{t_1}^{t_2} \|\dot{\gamma}(u)\| \, du \, .$$

When first exposed to differential calculus, it is common to apply it as a tool to find stationary points of functions, which are then examined to find local and global minima and maxima. Likewise, we care about finding and examining stationary points of of functionals to get minimal and maximal functions, which are called extremals.

Definition 2. Let S be some functional. A function f is an *extremal* of S if $\Delta S = S[f] - S[g]$ has the same sign for all g in a neighborhood of f.

There will be more focus on finding minimal extremals rather than maximal extremals in this paper, as this is what the problems we will examine are concerned with.

3. The Euler-Lagrange Equation

We follow Courant and Hilbert's proof of the one-dimensional Euler-Lagrange equation in [1]; to do so, we need a lemma.

Lemma 3 (The fundamental lemma of calculus of variations)**.** *Let f be continuous on (a, b). If*

$$\int_a^b f(x)h(x) \, dx = 0$$

for all smooth h on (a, b) where $h(a) = h(b) = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

A short proof of this lemma can be found in [2]. We can now prove the Euler-Lagrange equation.

Theorem 4 (The Euler-Lagrange equation)**.** *Let*

$$S[f] = \int_{x_0}^{x_1} L(x, f(x), f'(x)) \, dx,$$

where x_0, x_1 are constants and both $f(x)$ and $L(x, f(x), f'(x))$ are twice continuously differentiable with respect to all their arguments. Then f is a stationary point of S if and only if

$$\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0.$$

Proof. Taking any extremal f of S, a slight nudge within the boundary of integration can be written as $f + \varepsilon\mu$, where $\varepsilon > 0$ is small, μ is differentiable, and both $\mu(x_0) = \mu(x_1) = 0$. ($\varepsilon\mu$ is called the *variation* of f and can be also written as δf.) We define

$$\Phi(\varepsilon) = S[f + \varepsilon\mu] = \int_{x_0}^{x_1} L(x, f(x) + \varepsilon\mu(x), f'(x) + \varepsilon\mu'(x)) \, dx,$$

and investigate $\Phi'(\varepsilon)$:

$$\begin{aligned}\frac{d\Phi}{d\varepsilon} =& \frac{d}{d\varepsilon}\int_{x_0}^{x_1} L(x, f(x)+\varepsilon\mu(x), f'(x)+\varepsilon\mu'(x))\,dx \\ =& \int_{x_0}^{x_1} \frac{d}{d\varepsilon} L(x, f(x)+\varepsilon\mu(x), f'(x)+\varepsilon\mu'(x))\,dx \\ =& \int_{x_0}^{x_1} (\mu(x)\frac{\partial L}{\partial f}(x, f(x)+\varepsilon\mu(x), f'(x)+\varepsilon\mu'(x)) \\ &+ \mu'(x)\frac{\partial L}{\partial f'}(x, f(x)+\varepsilon\mu(x), f'(x)+\varepsilon\mu'(x)))\,dx\end{aligned}$$

Notice that when $\varepsilon = 0$, we get an extremum for Φ which we can integrate by parts:

$$\begin{aligned}0 &= \int_{x_0}^{x_1} (\mu(x)\frac{\partial L}{\partial f}(x, f(x), f'(x)) + \mu'(x)\frac{\partial L}{\partial f'}(x, f(x), f'(x)))\,dx \\ &= \int_{x_0}^{x_1} (\frac{\partial L}{\partial f}(x, f(x), f'(x)) - \frac{d}{dx}\frac{\partial L}{\partial f'}(x, f(x), f'(x)))\mu(x)\,dx + (\mu(x)\frac{\partial L}{\partial f'}(x, f(x), f'(x)))\Big|_a^b \\ &= \int_{x_0}^{x_1} (\frac{\partial L}{\partial f}(x, f(x), f'(x)) - \frac{d}{dx}\frac{\partial L}{\partial f'}(x, f(x), f'(x)))\mu(x)\,dx\end{aligned}$$

since $\mu(a) = \mu(b) = 0$. Applying the fundamental lemma of calculus of variations, we complete the proof. ■

It should be emphasized that on its own, the Euler-Lagrange equation is necessary but not sufficient criteria for finding an extremal; however, it is an extremely useful tool for finding candidates for extremals, and it is the one we will use for the upcoming problems. We can think of it as the first derivative test for functionals.

Definition 5. $\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'}$ is the *functional derivative* of $S[f]$, and can be written as $\delta S[f]$.

Those with knowledge in physics might recognize $L(x, f, f')$ as the Lagrangian, which is the kinetic energy minus the potential energy in non-relativistic cases. Finding stationary points of the integral of the Lagrangian, which is the action functional, generates the laws of a system, and many developments came from the ingenuity to do that with a new system.

Now that we have the Euler-Lagrange equation, we should try using it to get a feel for how it is in practice with a problem. Among the simplest problems we can apply it to is to find the shortest curve connecting two points — we should end up with the equation for a straight line. Initially, we run into an issue: in differential geometry, we consider curves as parameterizations of some other variable, while the Euler-Lagrange equation was solved such that y is a function of x. However, we can get around this issue without much trouble.

Example. We know that the arclength in $\mathbb{R}^2$ from t_1 to t_2 is the functional

$$\gamma \mapsto S[\gamma] = \int_{t_1}^{t_2} \|\dot{\gamma}(u)\|\,du = \int_{t_1}^{t_2} \sqrt{x'(u)^2 + y'(u)^2}\,du\,.$$

What we can do is take each component of γ and individually apply the Euler-Lagrange equation, and then combine the results together to find our extremal. So, the equations we're working with are

$$L = \sqrt{x'(u)^2 + y'(u)^2}$$
$$0 = \frac{\partial L}{\partial x} - \frac{d}{du}\frac{\partial L}{\partial x'}$$
$$0 = \frac{\partial L}{\partial y} - \frac{d}{du}\frac{\partial L}{\partial y'}.$$

Notice that the first term vanishes in both Euler-Lagrange equations, so we simply differentiate L with respect to x' and y' respectively, set the expression equal to zero, and solve for x' and y'. We obtain

$$x' = \pm\frac{y'}{\sqrt{1-a}} \implies x = \pm\frac{1}{\sqrt{1-a}}y + c$$
$$y' = \pm\frac{x'}{\sqrt{1-b}} \implies y = \pm\frac{1}{\sqrt{1-b}}x + d$$

with constants a, b, c, d, integrating with respect to u. These are both equations for lines, so the shortest curve connecting two points is a straight line as expected.

In this example, just one of the two equations would have sufficed to get our answer, but we should not assume this will always be the case. However, when it is, we can just change the variable of integration and arrive at the same result without needing to do this. The justification for us applying the Euler-Lagrange equation in this way is broadly the same as the proof of Theorem 4, with some extra care in our usage of the fundamental lemma to separate the expressions into multiple equations.

4. The Brachistochrone

As was briefly mentioned in the introduction, the brachistochrone problem played a large role in the development of calculus of variations — its historical significance and its relative fame among the field's problems make it only right for us to solve it in this paper.

Theorem 6 (The brachistochrone)**.** *Let γ be a curve connecting the origin and the point $\gamma(t_1) = (x_1, y_1)$ where $x \geq 0$ and $y \leq 0$, such that the time taken for a point mass acted on by gravity alone to move from the origin to (x_1, y_1) is minimized. Then γ is a cycloid.*

Proof. We start by again considering the arclength

$$S[\gamma] = \int_0^{t_1} \|\dot{\gamma}(u)\|\, du = \int_0^{t_1} \sqrt{x'(u)^2 + y'(u)^2}\, du\,.$$

Notice that $S[\gamma]$ is the distance traveled by our point mass, and that we can minimize the time taken by finding a minimal extremal of a functional of time $T[\gamma]$. We have $t = \frac{d}{v}$, so all we need now is to find what v is. We can use another fact from physics, that being the conservation of the quantity $\frac{1}{2}mv^2 + mgh$ with $m, g, h \geq 0$; since $y \leq 0$, we can say $y = -h$ and solve for v, which gives $y = \sqrt{2gy}$. Now, integrating in terms of y instead, we can consider

$$T[\gamma] = \int_0^{y_1} \sqrt{\frac{x'(y)^2 + 1}{2gy}}\, dy\,.$$

Since we're trying to find an extremal, a constant factor doesn't affect anything, so we can discard the $\frac{1}{\sqrt{2g}}$. Applying the Euler-Lagrange equation, we have

$$\begin{aligned} \frac{\partial L}{\partial x} - \frac{d}{dy}\frac{\partial L}{\partial x'} &= 0 \\ \implies \frac{d}{dy}\frac{x'(y)}{\sqrt{yx'(y)^2 + y}} &= 0. \end{aligned}$$

Integrating, rearranging, and integrating again gives

$$x(y) = \int \sqrt{\frac{y}{2r - y}}\, dy$$

for some constant r, and this is our curve. We can see what the curve is with the substitution $y(u) = r(1 - \cos u)$, which gives

$$x(u) = \int r\sqrt{\frac{(1 - \cos u)(1 - \cos^2 u)}{(1 + \cos u)}}\, du = r(u - \sin u),$$

with no constant since we start at the origin; this is a common parametrization of a cycloid, showing that the brachistochrone is a cycloid. ■

An interesting fact is that the cycloid is also the tautochrone, meaning regardless of where the point mass begins, it will reach the bottom of the curve in the same amount of time. At least one of the initial solvers of the brachistochrone problem explicitly noted this equivalence, as the tautochrone was solved first — but its analytical solution came after the brachistochrone's.

We will not prove the tautochrone, but one way to prove it is based on viewing the curve as a simple harmonic oscillator. Treating a pendulum as an example of one, regardless of where you start the swing from, the period of the pendulum will be the same, and this directly translates to the time spent rolling on the tautochrone.

5. Minimal Surfaces

We will now briefly look at minimal surfaces. Recall that minimal surfaces are those with a constant mean curvature of zero — a more physically-interpretable definition is that for any point on the surface, the area of its neighborhood is minimized. This leads to a definition in the language of calculus of variations:

Definition 7. A surface is *minimal* if and only if it is an extremal of the area functional.

One generalization of the Euler-Lagrange equation is theoretically applicable to minimal surfaces.

Theorem 8 (A generalized Euler-Lagrange equation)**.** *Let*

$$S[y] = \int_{\mathbf{\Omega}} L(x_1, x_2, \ldots, x_n, f, f_1, f_2, \ldots, f_n)\mathrm{d}\mathbf{x},$$

where $\mathbf{\Omega}$ *is some surface,* $f_i = \frac{\partial f}{\partial x_i}$ *and both* f *and* L *are twice continuously differentiable with respect to all their arguments. Then* f *is a stationary point of* S *if and only if*

$$\frac{\partial L}{\partial f} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial L}{\partial f_i} = 0.$$

If $S[f]$ is an energy functional and $n = 2$, then the equation represents soap films, which are one way that minimal surfaces manifest in reality — soap creates surfaces that require the least energy.

Applied to minimal surfaces, the Euler-Lagrange equation is far too hard to work with; the renaissance of minimal surfaces came when other methods were discovered — but, that can be taken as evidence of how widely applicable calculus of variations is.

References

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