LIE GROUPS

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1. Introduction

In this paper, I will give a brief introduction to notions of group theory necessary for the study of Lie groups, define and explore a few interesting examples of them, and define Lie algebras and highlight their connection to Lie groups. I will also introduce a few properties that will not be explored, as to encourage and point the reader towards further study. This paper is intended as a jumping-off point for those interested in Lie groups and Lie theory, but with otherwise little to no background in it.

2. A brief journey into group theory

In order to start inspecting Lie groups, we must first familiarize ourselves with groups and other notions from abstract algebra.
We will first give a brief definition of what a binary operation is:

Definition 2.1. A binary operation on a nonempty set A is a function $\star : A \times A \to A$ such that for all $a, b \in A$, $a \star b$ is defined.

Informally, groups are a way to encode symmetry and geometric transformations. Formally, they are defined as:

Definition 2.2. A group is a nonempty set G together with a binary operation $\cdot : G \times G \to G$ such that it satisfies the following axioms:

(1) **Associativity**: For all $a, b \in G$ we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
(2) **Identity element**: There exists an element $e \in G$ such that for all $a \in G$ we have $a \cdot e = e \cdot a = a$;
(3) **Inverse element**: For all $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

In addition, if $\cdot$ is commutative, i.e., for all $a, b \in G$, $a \cdot b = b \cdot a$, we call it an abelian or commutative group. We use the notation $(G, \cdot)$ for a group with the operation $\cdot$, although if the operation is implicit, we may just write G.

Example. $(\mathbb{R}^n, +)$ is an abelian group where $+$ is the addition of vectors.

Example. We define $U = \{z \in \mathbb{C} | \, |z| = 1\}$ as the units of the complex numbers. If $\cdot$ is the multiplication of complex numbers, then $(U, \cdot)$ is an abelian group. We call it the unit circle group, since it can be represented as the unit circle.

Example. We define $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) \neq 0\}$ be the set of all invertible real matrices. Then with $\cdot$ being matrix multiplication we get the group $(GL_n(\mathbb{R}), \cdot)$. We call it the general linear group of degree n.

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Now we shall take a look at a few objects useful in the study of groups.

Definition 2.3. Let $(G, \cdot)$ be a group, a nonempty subset $H \subseteq G$ is a *subgroup* of G if the following conditions are met:

(1) For all $x, y \in H$, $x \cdot y \in H$;
(2) If $x \in H$, then $x^{-1} \in H$.

In other words, H is a subgroup of G if $(H, \cdot|_H)$ is a group, where $\cdot|_H$ is the restriction of $\cdot$ to H. We denote this relation as $H \leq G$, read as H is a subgroup of G. We call H a *proper subgroup* if it is not the trivial subgroup $\{e\}$ or G itself. We denote this as $H < G$.

Example. We define the set $SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) | \det(A) = 1\}$. It is a subgroup of $GL_n(\mathbb{R})$ with matrix multiplication, and we call the group $(SL_n(\mathbb{R}), \cdot)$ the special linear group of degree n.

Another notation shorthand we shall use is writing $x \cdot y$ as xy. This is the *multiplicative* notation of groups.

Definition 2.4. Let H be a subgroup of G. Then we define:

(1) The *left coset* of H as $gH = \{h \in H |\ gh\}$ for a $g \in G$;
(2) The *right coset* of H as $Hg = \{h \in H |\ hg\}$ for a $g \in G$.

We can think of a coset as a "transposition" of H by g. Since in general groups are not commutative, in general $gH \neq Hg$. But for certain subgroups, it is true that $gH = Hg$ for all $g \in G$. We call these special subgroups:

Definition 2.5. A *normal subgroup* N of G is a subgroup such that $gN = Ng$ for all $g \in G$. This is equivalent to the statement $N = gNg^{-1} = \{n \in N |\ gng^{-1}\}$ for all $g \in G$. We denote this as $N \lhd G$.

Definition 2.6. Let $(G, \cdot)$ be a group. The set $Z(G) = \{x \in G |\ x \cdot y = y \cdot x \text{ for all } y \in G\}$ is called the *center* of G.

It is clear that a group is abelian if and only if $Z(G) = G$. Thus one can think of $Z(G)$ as a sort of measure of how abelian a group is. $Z(G)$ is also never the empty set because the identity element e commutes with all other elements.

Definition 2.7. We say that a group G is simple if and only if its only normal subgroups are $\{e\}$ and itself.

It is quite clear that for G to be simple and non-abelian, $Z(G) = \{e\}$ i.e it needs to include only the identity element. If there were an $x \in Z(G), x \neq e$ then we can take the set of powers of x, $\langle x \rangle = \{k \in \mathbb{Z} |\ x^k\} \neq G$ and it is clear that for any $g \in G$, $g\langle x \rangle = \langle x \rangle g$ and thus $\langle x \rangle \lhd G$, so G cannot be simple.

Definition 2.8. Let N be a normal subgroup of $(G, \cdot)$. We define the set $G/N = \{g \in G |\ gN\}$ as the set of all left cosets of N in G. If we define the binary operation $* : G/N \times G/N \to G/N$ as $g_1N * g_2N = (g_1 \cdot g_2)N$ for all $g_1, g_2 \in G$. Then we can define a group, called the *Quotient group* between G and N, as $(G/N, *)$.

Note: You get the same construction if you use the right coset instead. You can prove that it is necessary and sufficient for N to be a normal subgroup of G in order for $(G/N, *)$ to be a group. This is left as an exercise to the reader.

As we have often encountered, we are interested in functions between objects that preserve their structure. For example, in linear algebra, we are interested in linear transformations. In topology, we are interested in homomorphism. In group theory, we are interested in:

Definition 2.9. A *group homomorphism* f between the groups $(G_1, \cdot)$ and $(G_2, *)$ is a function $f : G_1 \to G_2$ such that:
(1) $f(e_1) = e_2$;
(2) $f(x \cdot y) = f(x) * f(y)$ for all $x, y \in G_1$;
(3) $f(x^{-1}) = (f(x))^{-1}$ for all $x \in G_1$.

If in addition f is bijective, we call f a *group isomorphism*. If there is a group isomorphism between the groups G_1 and G_2, then we say they are isomorphic, and denote this relation as $G_1 \cong G_2$.
A group homomorphism from G to itself is called a group endomorphism. If in addition, it is bijective, we call it a group automorphism.
We denote the set of all group endomorphisms and automorphisms of G as $\mathrm{End}(G)$ and $\mathrm{Aut}(G)$ respectively.

Definition 2.10. Let $(G_1, \cdot)$, $(G_2, *)$ be groups and $f : G_1 \to G_2$ a group homomorphism. We define the set $\ker(f) = \{g_1 \in G_1 | \ f(g_1) = e_2\}$. It is called the *kernel* of f.

Theorem 2.1. *[Noe27] Let G_1 and G_2 be group homomorphism, then, if we let $\mathrm{Im}(f)$ denote the image of f:*
(1) $\ker(f) \triangleleft G_1$;
(2) $\mathrm{Im}(f) \leq G_2$;
(3) $\mathrm{Im}(f) \cong G/\ker(f)$.

In particular, if f is surjective, we have $G_2 \cong G_1/\ker(f)$.

The proof is outside the scope of this paper. One can find it in any abstract algebra textbooks.

3. Lie groups

We now turn to the main topic of our discussion. In order to define lie groups, we must first define:

Definition 3.1. Let M be a set. A *chart* on M is a bijection of a subset $U \subset M$ onto an open subset of $\mathbb{R}^n$. We say the chart takes values in $\mathbb{R}^n$ or simply that the chart is $\mathbb{R}^n$-valued. A chart $f : U \to f(U)$ is traditionally indicated by the pair (U, f) and the pair itself is also called a chart.

Definition 3.2. An dimensional differentiable manifold of class C^r is a set M together with a specified C^r structure on M such that the topology induced by the C^r structure is *Hausdorff* and *paracompact*. If the charts are $\mathbb{R}^n$-valued then we say the manifold has dimension n.

If one wants to understand this definition better, I point towards [Mun00] for the necessary topological background and [Lee13] for a detailed discussion of manifolds.

We will deal with *smooth* manifolds i.e C^∞ class manifolds. Informally, one can view them as a generalization of surfaces to n dimensions instead of just 3.
It is now time to define:

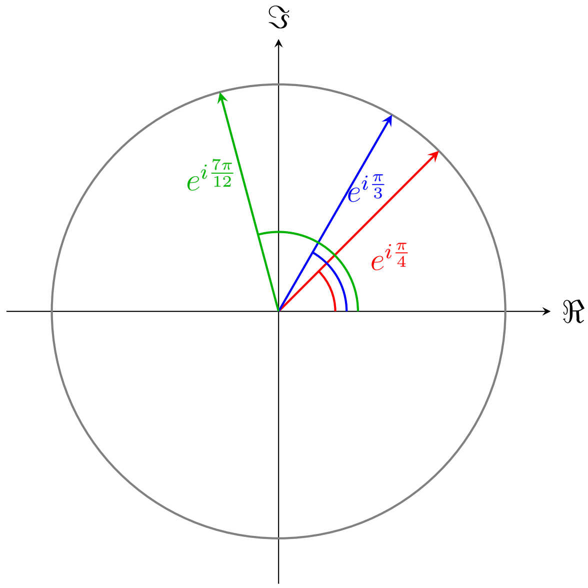

Figure 1. The rotations of the unit circle.

Definition 3.3. A smooth manifold G is a Lie group if it is a group and the maps $\mu : G \times G \to G$, $\mu(g_1, g_2) = g_1 g_2$ and $\nu : G \to G$, $\nu(g) = g^{-1}$ are smooth.

As groups in general are a way to represent symmetry, Lie groups represent symmetry that is also continuous. We can see this with a few examples of Lie groups, specifically those embodying **rotations**:

Example. If you recall, we defined the set U as the set of units in the complex plane, and they form a group under complex multiplication. From Euler's formula, we have $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, which clearly all members of U can be written as, since they are of length 1. From here we get that $e^{ix} \cdot e^{iy} = \cos(x+y) + i\sin(x+y)$. So if we represent it in the complex plane, we get that e^{ix} is rotated by y radians counterclockwise around the unit circle. If we take x to be $\frac{\pi}{4}$ and y to be $\frac{\pi}{3}$ we can draw it.
This is the Lie group of rotations of the unit circle, $SO(2) \cong U$. Note that it is an abelian group.

Example. Now we turn to $3D$ euclidean space. We shall look at the origin and rotations around it. A rotation around the origin in $\mathbb{R}^3$ is an orientation-preserving isometry of Euclidean space that leaves the origin fixed. If we define $r : \mathbb{R}^3 \to \mathbb{R}^3$ as a rotation, then $\|\mathbf{r}(\mathbf{x}) - \mathbf{r}(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^\mathbf{3}$. Since the origin is fixed, $\mathbf{r}(\mathbf{0}) = \mathbf{0}$ and $\|\mathbf{r}(\mathbf{x})\| = \|\mathbf{x}\|$ for all $x \in \mathbb{R}^3$. If we take $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ the standard basis vector for $\mathbb{R}^3$. Then:

$$\|\mathbf{r}(\mathbf{e_i})\| = \|\mathbf{e_i}\| = 1$$

Also if $i \neq j$:

$$\|\mathbf{r}(\mathbf{e_i}) - \mathbf{r}(\mathbf{e_j})\|^2 = \|\mathbf{e_i} - \mathbf{e_j}\|^2 = 2$$
$$\|\mathbf{r}(\mathbf{e_i}) - \mathbf{r}(\mathbf{e_j})\|^2 = \|\mathbf{r}(\mathbf{e_i})\|^2 + \|\mathbf{r}(\mathbf{e_j})\|^2 - 2\mathbf{r}(\mathbf{e_i})\mathbf{r}(\mathbf{e_j})$$

Since the norms are all 1, we get that $\mathbf{r}(\mathbf{e_i})\mathbf{r}(\mathbf{e_j}) = \mathbf{0}$ and for all $i \neq j$ so, the vectors $\{\mathbf{r}(\mathbf{e_1}), \mathbf{r}(\mathbf{e_2}), \mathbf{r}(\mathbf{e_3})\}$ form an orthonormal basis of $\mathbb{R}^3$ and thus the rotation can be represented

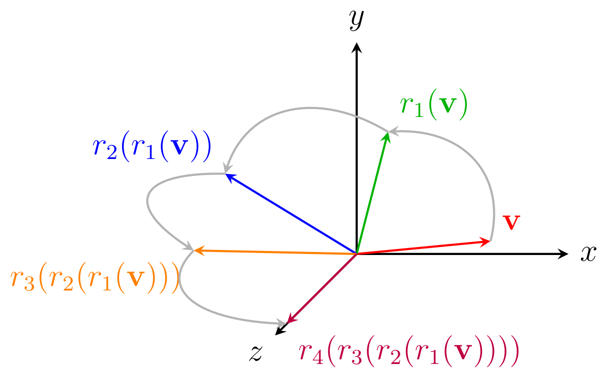

Figure 2. Composition of rotations in $\mathbb{R}^3$

as an orthogonal matrix i.e a matrix A with real entries such that A^T is its inverse. Due to it preserving orientation, this forces $\det(A) = 1$. Thus we can characterize the group of rotations around the origin in $\mathbb{R}^3$ as $SO(3) = \{A \in M_3(\mathbb{R}) |\ AA^T = A^TA = I_3$ and $\det(A) = 1\}$ with the operation being matrix multiplication. Note that it is not abelian, unlike $SO(2)$, since matrix multiplication is not generally commutative.
From this characterization, arises another Lie group! Notice that we didn't depend on being in $\mathbb{R}^3$, so we might be able to do a generalization in $\mathbb{R}^n$. In fact, if we drop the requirement for it to be orientation preserving and let it fix any point, we can describe the Lie group of all isometries that preserve a fixed point, called the *general orthogonal group*, $O_n = \{A \in M_n(\mathbb{R}) |\ AA^T = A^TA = I_n\}$. It is evident that $SO(3) < O_3$. Similarly, we get the group of rotations in $\mathbb{R}^4$ as $SO(4) = \{A \in M_4(\mathbb{R}) |\ AA^T = A^TA = I_4$ and $\det(A) = 1\} < O_4$. In general, the Lie group $SO(n) = \{A \in M_n(\mathbb{R}) |\ AA^T = A^TA = I_n$ and $\det(A) = 1\} < O_n$ is called the *special orthogonal group* or the *rotation group* of degree n.

Example. A *unitary* matrix $A \in M_n(\mathbb{C})$ is a matrix such that its conjugate transpose $A^* = \overline{A^T}$ is also its inverse. We define, with the operation of matrix multiplication, the Lie group $U(n) = \{A \in M_n(\mathbb{C}) |\ AA^* = A^*A = I_n\}$ called the *general unitary group.* As $\det(AB) = \det(A)\det(B)$ and the property of unitary matrices that $|\det(A)| = 1$, i.e. it lies in the unit circle, it can be verified that the map $\det : U_n \to U$ is a group homomorphism. From 2.1 we get $\ker(U(n)) \triangleleft U_n$. Obviously, $\ker(\det(U(n))) = \{A \in M_n(\mathbb{C}) |\ AA^* = A^*A = I_n$ and $\det(A) = 1\}$, which we define as the set $SU(n)$ called *the special unitary group.* If one is familiar with quaternions, a way to describe $SU(2)$ is by the fact that is isomorphic to the group of quaternions of norm 1, which means it is diffeomorphic to a $3-$sphere.

Example. The Lie group $\mathbb{T}^2 = U \times U$, where we define the group operation as $(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot a_2, b_1 \cdot b_2)$, is called the 2-dimensional torus. More generally, we define the k-th dimensional torus group is $\mathbb{T}^k = \underbrace{U \times U \times \cdots \times U}_{k \text{ times}}$ with the group operation defined similarly. We call it the torus group because it can be represented on a torus, as the intersection of two circles, one horizontal and one vertical, each an element of U.

The torus groups are quite important for studying the structure of Lie groups along with a few other notions. I will give their definitions, as they are important, but we shall not dwell on them, as they have little direct connections with notions from differential geometry

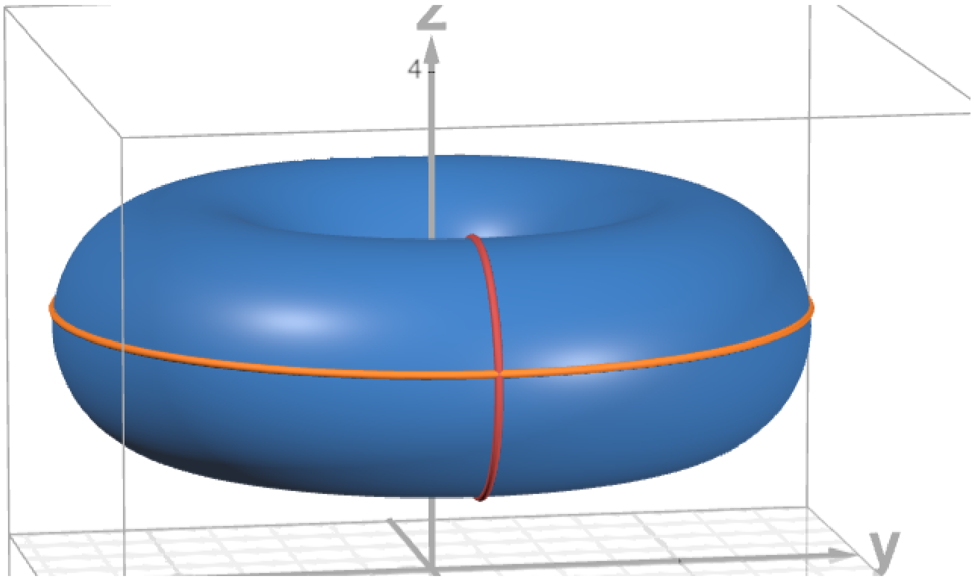

Figure 3. A point on the torus represented as the intersection of 2 circles.

and are thus outside the scope of this paper. I do, however, encourage the reader to research them, as there are some amazing results, like the complete classification of simple Lie groups.

Definition 3.4. The *maximal tori* of a Lie group $(G, \cdot)$ is a subgroup H such that $H \cong \mathbb{T}^k$ where k is maximal.

Definition 3.5. A Lie group $(G, \cdot)$ is path connected if G is path-connected. A topological space S is path connected if for any two points $x, y \in S$, there exists a continuous function $\gamma : [0, 1] \to S$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

A way to view this in terms of differential geometry is that you can draw a curve between any two points on the surface. An interesting result that that is not immediately obvious is:

Proposition 3.1. *[Sti08] For any n, $SO(n)$ is path-connected.*

Proof. For $n = 2$, we have the circle $SO(2)$, which is obviously path-connected. We shall use an induction argument. Now suppose $SO(n-1)$ is path-connected and let $A \in SO(n)$. It is sufficient to prove there is a path from I_n to A, since if there is a path from I_n to A and B, then there is a path from A to B.
Then we have to find a continuous motion taking the basis vector $\{\mathbf{e_1}, \mathbf{e_2}, \dots, \mathbf{e_n}\}$ to the final positions after the transformation, $A\mathbf{e_1}, A\mathbf{e_2}, \dots, A\mathbf{e_n}$. If $\mathbf{e_1}$ and $A\mathbf{e_1}$ are distinct(we can assume they are, since if not we can look at $\mathbf{e_2}$ and so on), they define a plane P, so by the path connectedness of $SO(2)$ we can move $\mathbf{e_1}$ to $A\mathbf{e_1}$ continuously via a rotation R of P. It then suffices to move continuously $R\mathbf{e_2}, R\mathbf{e_3}, \dots, R\mathbf{e_n}$ to $A\mathbf{e_2}, A\mathbf{e_3}, \dots, A\mathbf{e_n}$ respectively, keeping $A\mathbf{e_1}$ fixed.
Notice that since A and R are rotations, they preserve the orthogonality, therefore $R\mathbf{e_i}$ is orthogonal to $R\mathbf{e_1} = A\mathbf{e_1}$ and $A\mathbf{e_i}$ is orthogonal to $A\mathbf{e_1}$ for all $i \in \mathbb{N}, 1 < i \leq n$. Thus the required motion can take place in the space $\mathbb{R}^{n-1}$ orthogonal vectors to $A\mathbf{e_1}$, which exists by the assumption of path-connectedness of $SO(n-1)$ and is given by performing A and R in succession.
By induction we get that $SO(n)$ is path connected for all $n \in N$ ∎

Definition 3.6. A Lie group $(G, \cdot)$ is simple if it is algebraically a simple group.

Note: There is a disagreement in the literature on what the exact definition of a simple Lie group is, since many authors also require for the Lie group to be path-connected.

Definition 3.7. A Lie group $(G, \cdot)$ is *simply connected* if the topological space G is simply connected. A topological space S is simply connected if it is path connected and any loop γ, i.e a path γ at x_0 such that $\gamma(0) = \gamma(1) = x_0$ can be shrunken down to a point. More explicitly, if there exists a continuous function $H : [0,1] \times [0,1] \to S$ such that

(1) $H(s, 0) = \gamma(s)$ for all $s \in [0, 1]$;
(2) $H(s, 1) = x_0$ for all $s \in [0, 1]$;
(3) $H(0, t) = H(1, t) = x_0$ for all $t \in [0, 1]$.

Informally, this implies that the space has no holes in it. For example $\mathbb{T}^2$ is not simply connected but $SU(2)$ is.

4. Lie algebras

Often in math, it is useful to define auxiliary objects in order to study the one of interest. Lie groups are no different.

Definition 4.1. A Lie algebra is a vector space $\mathfrak{g}$ over a field F together with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, satisfying the following axioms:

- **Bilinearity:**
$$[ax + by, z] = a[x, z] + b[y, z], \quad [z, ax + by] = a[z, x] + b[z, y]$$
for all scalars $a, b \in F$ and all elements $x, y, z \in \mathfrak{g}$.
- **Alternating property:**
$$[x, x] = 0$$
for all $x \in \mathfrak{g}$.
- **Jacobi identity:**
$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$
for all $x, y, z \in \mathfrak{g}$.

Using bilinearity to expand the Lie bracket $[x+y, x+y]$ and using the alternating property shows that
$$[x, y] + [y, x] = 0$$
for all $x, y \in \mathfrak{g}$. Thus, bilinearity and the alternating property together imply
Anticommutativity:
$$[x, y] = -[y, x],$$
for all $x, y \in \mathfrak{g}$. If the field does not have characteristic 2, then anticommutativity implies the alternating property, since it implies
$$[x, x] = -[x, x].$$

Example. A vector space V can become a Lie algebra with the Lie bracket $[X, Y] = XY - YX$ for all $X, Y \in V$. This Lie bracket is called *the commutator.*

5. The connection between Lie algebras and Lie groups

There's a very interesting connection between the Lie algebras and Lie groups. Recall that Lie groups are defined over smooth manifolds. If we let our Lie group $(G, \cdot)$, and we consider a point $\mathbf{p} \in G$ and a curve γ on G that passes through $\mathbf{p}$, and consider its tangent vector $\dot{\gamma}$ at point $\mathbf{p}$. Similarly to the tangent plane, we define an analogue in higher dimension, the tangent space.

Definition 5.1. Let M be a smooth manifold and $\mathbf{p} \in M$ a point on it. The tangent space at $\mathbf{p}$, denoted $T_{\mathbf{p}}M$ is the vector space of all tangent vectors to all curves γ passing through $\mathbf{p}$ at point $\mathbf{p}$.

Note: This definition is simplified. If one wants to understand the full nuances of it, check [Lee13].
We can now create a correspondence between a Lie algebra and a Lie group as follows. This is sometimes given as an alternative definition.

Definition 5.2. Let $(G, \cdot)$ be a Lie group. The corresponding Lie algebra is the Lie algebra $\mathfrak{g}= T_eG$. This is codified in notation as $\text{Lie}(G) = \mathfrak{g}$.

Defining the Lie bracket in the general sense for the Lie algebras requires background beyond the scope of this paper.If one wants to learn and see the concrete definition, check [Hal15]. But if we restrict ourselves to matrix Lie groups, it is actually the commutator!
Another way to view the connection between them is through the exponential map.

Definition 5.3. Let $(G, \cdot)$ be a Lie group. A curve $\gamma : \mathbb{R} \to G$ is called a one-parameter subgroup if it satisfies the condition $\gamma(t)\gamma(s) = \gamma(t + s)$ for all $s, t \in \mathbb{R}$.

Definition 5.4. Let $(G, \cdot)$ be a Lie group and $\mathfrak{g}$ its corresponding Lie algebra. The exponential is a function $\exp : \mathfrak{g} \to G$ where the exponential of $X \in \mathfrak{g}$ is given by $\exp(X) = \gamma(1)$ where $\gamma : \mathbb{R} \to G$ is the unique one-parameter subgroup of G whose tangent vector at identity is equal to X.

From the chain rule, you easily get $\exp(tX) = \gamma(t)$. Therefore we can get a few other properties, like $\exp((t + s)X) = \exp(tX) \cdot \exp(sX)$ and $\exp(-X) = (\exp(X))^{-1}$,as well as if X, Y commute then $\exp(X + Y) = \exp(X) \cdot \exp(Y)$.
In the case of matrix Lie algebras, we get a very explicit form for the exponential map, $\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} = I_n + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \dots$ which is very similar to the Taylor expansion of e^x.

Example. Let us take the Lie group $(GL_n(\mathbb{R}), \cdot)$. Then its lie algebra is $\mathfrak{gl_n} = M_n(\mathbb{R})$ with the Lie bracket $[X, Y] = XY - YX$. and the exponential map as above.

Example. For the torus group $\mathbb{T}^2$, we can actually visualize it and its tangent plane! Check figure 4. Its exponential map is given by $\exp(X) = (e^{it_1}, e^{it_2})$.

I will end this paper with two theorems, whose proofs are significantly beyond the scope of this paper, but nonetheless I find extremely beautiful and that showcase the powerful connection between the two concepts.

Theorem 5.1. *(Lie's third theorem) [Hal15] Every finite-dimensional real Lie algebra is the Lie algebra of some simply connected Lie group.*

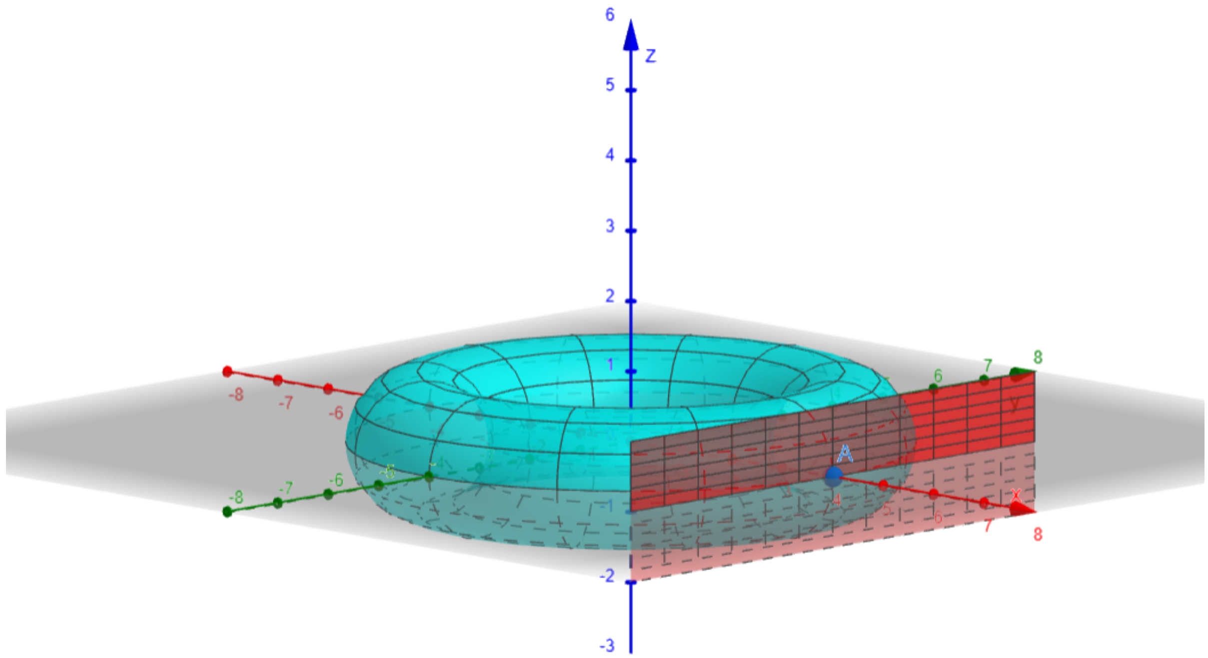

Figure 4. $\mathbb{T}^2$, the identity element and the tangent plane

In short, you can find a smooth manifold with no holes such that its tangent space at identity is your Lie algebra. It's extremely powerful to not only know that whatever Lie algebra you find, there is a corresponding Lie group, but also that the manifold it is defined on has no holes!

Theorem 5.2. *(The homomorphisms theorem) [Hal15] If* $\phi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ *is a Lie algebra homomorphism and if* G *is simply connected, then there exists a (unique) Lie group homomorphism* $f : G \to H$ *such that* $\phi = \mathrm{d}f$*.*

This again, by way of Lie algebras, gives you tools to investigate the underlying Lie groups.

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