Minimal Surfaces

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Abstract

In this paper, we examine the theory of minimal surfaces. These are surfaces that have locally minimized area and are equivalently characterized by having vanishing (0) mean curvature. In this paper, we study their variational characterization, derive the minimal surface equation in detail, and discuss classical examples and results such as Bernstein's theorem and the maximum principle. We highlight their appearance in physical systems, complex analysis, and geometry, as well as their applications outside of mathematics, such as in materials science, biology, and design.

1 Introduction

The study of minimal surfaces lies at the intersection of geometry, analysis, and physics. These are surfaces in $\mathbb{R}^3$ that locally minimize area, making them the geometric analog of geodesics (which minimize length). Historically, minimal surfaces were first studied by Joseph-Louis Lagrange in the 18th century. He formulated the variational problem of finding the surface with the least area spanning over a given curve. Lagrange successfully derived the minimal surface equation (Euler–Lagrange equation) as an application of the calculus of variations. While Lagrange did not explicitly find solutions other than simple planes, his works laid the foundation for future research on minimal surfaces, allowing Leonhard Euler and Jean Baptiste Meusnier to later discover specific minimal surfaces such as the catenoid and helicoid.

Minimal surfaces appear widely in the natural world, arising wherever systems tend to minimize energy subject to geometric constraints. The most famous example is the behavior of soap films. When a wire frame is dipped in a soap solution, the resulting soap film spans the frame by adopting a shape that minimizes surface area. This phenomenon is the result of surface tension, a physical force that drives the film to assume a configuration of least energy. Because surface area corresponds to the surface energy in such systems, the resulting shape is a surface that locally minimizes the area. These surfaces provide real-world illustrations of the calculus of variations: they are critical points of the area functional under fixed boundary conditions. In engineering, minimal surfaces are used to design strong yet material-efficient structures, particularly in tensile architecture (e.g., tent roofs, cable nets). Therefore, the concept of a minimal surface is both mathematically fundamental and deeply rooted in the structure and behavior of the physical universe.

Mathematically, a surface is minimal if the mean curvature H vanishes everywhere. If κ_1 and κ_2 are the principal curvatures, then

$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$

Thus, a surface is minimal if it is equally curved in opposite directions at every point, as shown below.

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = 0$$
$$\kappa_1 = -\kappa_2$$

That is, in one direction, the surface bends by an amount κ_1, and in the perpendicular (principal) direction it bends by the exact opposite amount $-\kappa_1$. The surface curves are equal in both directions, but the average curvatures cancel out, resulting in a zero mean curvature.

The topic of minimal surfaces has expanded over the centuries and includes areas such as global analysis, complex variables, differential equations, and geometric measurement theory. In this paper, we explore the core ideas and provide detailed derivations, examples, as well as applications both in and outside of mathematics.

2 The Variational Characterization

Let $\vec{X} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth parametrization of a surface, where U is an open domain with coordinates (u, v). The *area functional* assigns to $\vec{X}$ the total surface area:

$$A[\vec{X}] = \iint_U \|\vec{X}_u \times \vec{X}_v\| \, du \, dv.$$

In the calculus of variations, we study how $A[\vec{X}]$ changes when the surface is slightly deformed. We introduce a *variation* of $\vec{X}$:

$$\vec{X}_t(u, v) = \vec{X}(u, v) + t \, \vec{V}(u, v),$$

where $\vec{V}$ is a smooth vector field with compact support in U, and t is a real parameter. The *first variation* of area is the derivative of $A[\vec{X}_t]$ at $t = 0$:

$$\delta A[\vec{X}] = \frac{d}{dt} A[\vec{X}_t] \bigg|_{t=0}.$$

This is just an ordinary derivative computation, but applied to a functional (a function of a function).

Carrying out the differentiation under the integral sign and using vector calculus identities, one finds:

$$\delta A = -\iint_U 2H \, \langle \vec{V}, \vec{N} \rangle \, dA,$$

where H is the mean curvature and $\vec{N}$ is the unit normal.

Since $\vec{V}$ is arbitrary, the only way δA can vanish for *every* variation is if $H \equiv 0$. This proves:

Theorem 1. *A smooth surface $\vec{X}$ is minimal if and only if its mean curvature vanishes identically.*

For the special case where the surface is given as a graph $z = f(x, y)$, the area functional becomes:

$$A[f] = \iint_\Omega \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy.$$

This is now an integral functional depending on f and its first derivatives. The *Euler–Lagrange equation* from the calculus of variations gives the condition for $A[f]$ to be stationary:

$$\frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{1 + f_x^2 + f_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{f_y}{\sqrt{1 + f_x^2 + f_y^2}}\right) = 0.$$

This partial differential equation is the *minimal surface equation*, and its solutions are exactly the graphs with zero mean curvature.

3 Minimal Surface Equation

If a surface is given as the graph $z = f(x, y)$, the area becomes:

$$A[f] = \iint_\Omega \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy.$$

Applying the Euler–Lagrange equation to this functional yields:

$$\frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{1 + f_x^2 + f_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{f_y}{\sqrt{1 + f_x^2 + f_y^2}}\right) = 0.$$

Multiplying through by the denominator and simplifying yields the classical form:

$$(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = 0.$$

This nonlinear partial differential equation is known as the minimal surface equation in the differential equation definition. Solutions to it yield graphs of minimal surfaces.

4 Classical Examples

Plane

A plane is the simplest example of a minimal surface, where $f(x, y) = ax + by + c$. In this case, $f_{xx} = f_{yy} = f_{xy} = 0$, so the minimal surface equation is trivially satisfied.

Catenoid

A Catenoid is generated by rotating a catenary around an axis:

$$\vec{X}(u,v) = (c\cosh(\frac{v}{c})\cos u, c\cosh(\frac{v}{c}) sinu, v).$$

where $u \in [-\pi, \pi)$, $v \in \mathbb{R}$, and c is a non-zero real constant.

4.0.1 Proof

Consider the parametrization of the catenoid

$$\vec{X}(u,v) = \big(c\cosh(\tfrac{v}{c})\cos u,\ c\cosh(\tfrac{v}{c})\sin u,\ v\big), \qquad u \in [-\pi,\pi),\ v \in \mathbb{R},\ c \neq 0.$$

We verify directly that its mean curvature vanishes.

Compute first derivatives:

$$\vec{X}_u = (-c\cosh(\tfrac{v}{c})\sin u,\ c\cosh(\tfrac{v}{c})\cos u,\ 0), \qquad \vec{X}_v = (\sinh(\tfrac{v}{c})\cos u,\ \sinh(\tfrac{v}{c})\sin u,\ 1).$$

Compute the first fundamental form coefficients:

$$E = \langle \vec{X}_u, \vec{X}_u \rangle = c^2\cosh^2\big(\tfrac{v}{c}\big), \qquad F = \langle \vec{X}_u, \vec{X}_v \rangle = 0, \qquad G = \langle \vec{X}_v, \vec{X}_v \rangle = \sinh^2\big(\tfrac{v}{c}\big) + 1 = \cosh^2\big(\tfrac{v}{c}\big).$$

The normal (up to orientation) is obtained from the cross product

$$\vec{X}_u \times \vec{X}_v = c\cosh\big(\tfrac{v}{c}\big)\big(\cos u,\ \sin u,\ -\sinh(\tfrac{v}{c})\big),$$

So the unit normal is

$$\vec{N} = \frac{\vec{X}_u \times \vec{X}_v}{\|\vec{X}_u \times \vec{X}_v\|} = \big(\operatorname{sech}(\tfrac{v}{c})\cos u,\ \operatorname{sech}(\tfrac{v}{c})\sin u,\ -\tanh(\tfrac{v}{c})\big).$$

Compute second derivatives needed for the second fundamental form:

$$\vec{X}_{uu} = -c\cosh\big(\tfrac{v}{c}\big)(\cos u,\ \sin u,\ 0),$$

$$\vec{X}_{uv} = \sinh\big(\tfrac{v}{c}\big)(-\sin u,\ \cos u,\ 0), \qquad \vec{X}_{vv} = \tfrac{1}{c}\cosh\big(\tfrac{v}{c}\big)(\cos u,\ \sin u,\ 0).$$

Now the second fundamental form coefficients are

$$e = \langle \vec{X}_{uu}, \vec{N} \rangle = -c\cosh\big(\tfrac{v}{c}\big)\operatorname{sech}\big(\tfrac{v}{c}\big) = -c,$$

$$f = \langle \vec{X}_{uv}, \vec{N} \rangle = 0, \qquad g = \langle \vec{X}_{vv}, \vec{N} \rangle = \tfrac{1}{c}\cosh\big(\tfrac{v}{c}\big)\operatorname{sech}\big(\tfrac{v}{c}\big) = \tfrac{1}{c}.$$

The mean curvature H is given by

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{eG + gE}{2EG} = \frac{(-c)\cosh^2(\frac{v}{c}) + \frac{1}{c}c^2\cosh^2(\frac{v}{c})}{2c^2\cosh^4(\frac{v}{c})} = 0.$$

Thus $H \equiv 0$ for the catenoid, so the catenoid is a minimal surface.

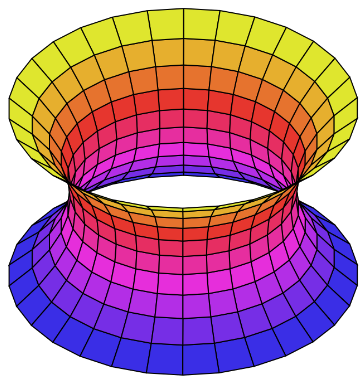

Figure 1: Image of a Catenoid

Helicoid

A Helicoid is generated by moving a straight line along a helical path:

$$\vec{X}(u, v) = (v \cos u, v \sin u, cu).$$

where u, $v \in \mathbb{R}$, and c is a constant.

This surface is ruled (i.e., composed of straight lines) and is still minimal.

4.0.2 Proof

Consider the parametrization of the helicoid

$$\vec{X}(u, v) = \big(v \cos u,\ v \sin u,\ cu\big), \qquad u, v \in \mathbb{R},\ c \neq 0.$$

We verify directly that its mean curvature vanishes.

Compute first derivatives:

$$\vec{X}_u = (-v \sin u,\ v \cos u,\ c), \qquad \vec{X}_v = (\cos u,\ \sin u,\ 0).$$

Compute the first fundamental form coefficients:

$$E = \langle \vec{X}_u, \vec{X}_u \rangle = v^2 + c^2, \qquad F = \langle \vec{X}_u, \vec{X}_v \rangle = 0, \qquad G = \langle \vec{X}_v, \vec{X}_v \rangle = 1.$$

The cross product is

$$\vec{X}_u \times \vec{X}_v = (-c \sin u,\ c \cos u,\ -v),$$

So the unit normal is

$$\vec{N} = \frac{\vec{X}_u \times \vec{X}_v}{\|\vec{X}_u \times \vec{X}_v\|} = \frac{(-c\sin u,\ c\cos u,\ -v)}{\sqrt{c^2+v^2}}.$$

Compute second derivatives:

$$\vec{X}_{uu} = (-v\cos u,\ -v\sin u,\ 0), \qquad \vec{X}_{uv} = (-\sin u,\ \cos u,\ 0), \qquad \vec{X}_{vv} = (0,\ 0,\ 0).$$

The second fundamental form coefficients are

$$e = \langle \vec{X}_{uu}, \vec{N} \rangle = 0, \qquad f = \langle \vec{X}_{uv}, \vec{N} \rangle = \frac{c}{\sqrt{c^2+v^2}}, \qquad g = \langle \vec{X}_{vv}, \vec{N} \rangle = 0.$$

The mean curvature H is

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{eG + gE}{2EG} = \frac{0 \cdot 1 + 0 \cdot (v^2 + c^2)}{2(v^2 + c^2)} = 0.$$

Thus $H \equiv 0$ for the helicoid, so the helicoid is a minimal surface.

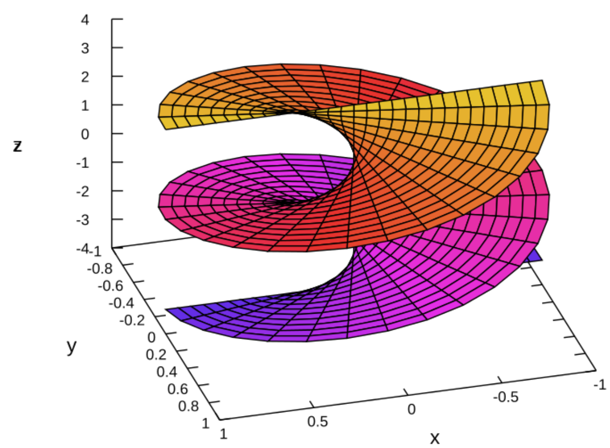

Figure 2: Image of a Helicoid with $-\pi \le u \le \pi$, $-1 \le v \le 1$, and $c = 1$.

Enneper's Surface

Enneper's Surface is defined by:

$$\vec{X}(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right).$$

Enneper's Surface is notable for its self-intersections and polynomial parameterization.

4.0.3 Proof

Consider the parametrization of Enneper's surface

$$\vec{X}(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ -v + \frac{v^3}{3} - u^2 v,\ u^2 - v^2\right), \quad u, v \in \mathbb{R}.$$

We verify directly that its mean curvature vanishes.

Compute first derivatives:

$$\vec{X}_u = \left(1 - u^2 + v^2,\ -2uv,\ 2u\right), \qquad \vec{X}_v = \left(2uv,\ -1 + v^2 - u^2,\ -2v\right).$$

Compute the first fundamental form coefficients:

$$E = \langle \vec{X}_u, \vec{X}_u \rangle = (1 - u^2 + v^2)^2 + (-2uv)^2 + (2u)^2,$$

$$F = \langle \vec{X}_u, \vec{X}_v \rangle = (1 - u^2 + v^2)(2uv) + (-2uv)(-1 + v^2 - u^2) + (2u)(-2v),$$

$$G = \langle \vec{X}_v, \vec{X}_v \rangle = (2uv)^2 + (-1 + v^2 - u^2)^2 + (-2v)^2.$$

A direct computation shows

$$E = G = (1 + u^2 + v^2)^2, \qquad F = 0.$$

The cross product is

$$\vec{X}_u \times \vec{X}_v = \left(-2u(1 + u^2 + v^2),\ -2v(1 + u^2 + v^2),\ (1 + u^2 + v^2)(1 - u^2 - v^2)\right),$$

So the unit normal is

$$\vec{N} = \frac{1}{1 + u^2 + v^2}\left(-2u,\ -2v,\ 1 - u^2 - v^2\right).$$

Compute second derivatives:

$$\vec{X}_{uu} = (-2u,\ -2v,\ 2), \quad \vec{X}_{uv} = (2v,\ -2u,\ 0), \quad \vec{X}_{vv} = (2u,\ 2v,\ -2).$$

The second fundamental form coefficients are

$$e = \langle \vec{X}_{uu}, \vec{N} \rangle = \frac{-2u(-2u) + (-2v)(-2v) + 2(1 - u^2 - v^2)}{1 + u^2 + v^2} = \frac{2 - 2u^2 - 2v^2 + 4u^2 + 4v^2}{1 + u^2 + v^2} = \frac{2(1 + u^2 + v^2)}{1 + u^2 + v^2}$$

$$f = \langle \vec{X}_{uv}, \vec{N} \rangle = \frac{(2v)(-2u) + (-2u)(-2v) + 0 \cdot (1 - u^2 - v^2)}{1 + u^2 + v^2} = 0,$$

$$g = \langle \vec{X}_{vv}, \vec{N} \rangle = \frac{(2u)(-2u) + (2v)(-2v) + (-2)(1 - u^2 - v^2)}{1 + u^2 + v^2} = \frac{-2 - 2u^2 - 2v^2}{1 + u^2 + v^2} = -2.$$

The mean curvature is

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = \frac{eG + gE}{2EG} = \frac{2(1 + u^2 + v^2)^2 + (-2)(1 + u^2 + v^2)^2}{2(1 + u^2 + v^2)^4} = 0.$$

Thus $H \equiv 0$ for Enneper's surface, so it is a minimal surface.

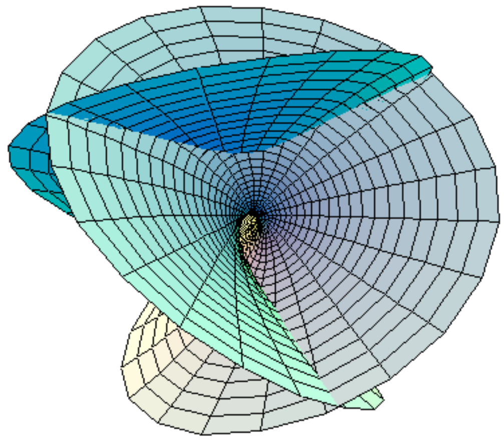

Figure 3: An Enneper's Surface, with intersection.

Scherk's Surface

Scherk's Surface was discovered in 1835 as a solution to the minimal surface equation:

$$z = \log\left(\frac{\cos y}{\cos x}\right), \quad |x|, |y| < \frac{\pi}{2}.$$

Scherk's Surface is periodic and doubly asymptotic.

4.0.4 Proof

Consider Scherk's surface defined implicitly by

$$z = \log\left(\frac{\cos y}{\cos x}\right), \quad |x|, |y| < \frac{\pi}{2}.$$

We verify that this surface satisfies the minimal surface equation.

The minimal surface equation for a graph $z = f(x, y)$ is

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0.$$

Compute first derivatives:

$$f_x = \frac{\partial z}{\partial x} = \frac{d}{dx}\log(\cos y) - \frac{d}{dx}\log(\cos x) = -\tan x,$$

$$f_y = \frac{\partial z}{\partial y} = \frac{d}{dy}\log(\cos y) - \frac{d}{dy}\log(\cos x) = \tan y.$$

Compute second derivatives:

$$f_{xx} = \frac{d}{dx}(-\tan x) = -\sec^2 x,$$

$$f_{yy} = \frac{d}{dy}(\tan y) = \sec^2 y,$$

$$f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}(-\tan x) = 0,$$

Since f_x depends only on x.

Substitute into the minimal surface equation:

$$(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = (1 + \tan^2 y)(-\sec^2 x) + (1 + \tan^2 x) \sec^2 y.$$

Recall the identity $1 + \tan^2 \theta = \sec^2 \theta$, so this becomes

$$\sec^2 y(-\sec^2 x) + \sec^2 x \sec^2 y = -\sec^2 x \sec^2 y + \sec^2 x \sec^2 y = 0.$$

Thus Scherk's surface satisfies the minimal surface equation and is therefore a minimal surface.

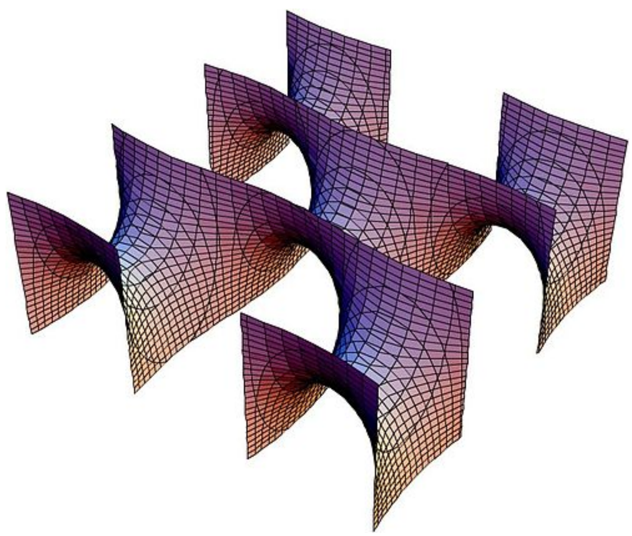

Figure 4: Five unit cells of Scherk's Surface placed together

5 Important Theorems

Bernstein's Theorem

In 1915, Sergei Bernstein proved that any entire minimal graph $f : \mathbb{R}^2 \to \mathbb{R}$ must be a plane. This means that if a surface defined by $z = f(x, y)$ is minimal and the domain is the whole $\mathbb{R}^2$, then f is an affine function. The theorem was later extended to higher dimensions, showing that the result still holds for $f : \mathbb{R}^n \to \mathbb{R}$ when $n \leq 7$. However, for $n \geq 8$, there exist counterexamples to the general case, showing that the property of all entire minimal graphs being planes is valid only in low dimensions.

For $n \geq 8$, a counterexample to Bernstein's theorem is given by the *Simons cone* in $\mathbb{R}^8$,

$$C = \{x \in \mathbb{R}^8 \ : \ x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}.$$

This cone is minimal, but not a hyperplane, showing that the statement of Bernstein's theorem fails in higher dimensions. It can be shown that C minimizes area in its homology class, and when expressed as a graph over $\mathbb{R}^7 \setminus \{0\}$, it gives a non-planar entire minimal graph in dimension $n = 8$.

Douglas–Rado Theorem

The Douglas–Rado theorem, proved by Jesse Douglas and Tibor Radó, solves Plateau's problem. It states that, given any simple closed curve $\gamma \subset \mathbb{R}^3$, there exists a minimal area surface bounded by γ. More precisely, there exists a minimal surface spanning γ that can be obtained as a solution to a variational problem for the area functional. This result established the existence of minimal surfaces for arbitrary Jordan curves in space and provided a rigorous foundation for studying physical soap films.

Maximum Principle

The Maximum Principle states that if two minimal surfaces intersect at an interior point and are tangent at that point, then they must coincide in a neighborhood of that point. In other words, two distinct minimal surfaces cannot touch at an interior point without crossing. Equivalently, a minimal surface cannot attain a strict local maximum or minimum for its height function at an interior point unless the surface is a plane. This principle is important for proving uniqueness results for solutions of the minimal surface equation.

6 Applications of Minimal Surfaces

- **Geometry and Topology:** Minimal surfaces are used to study the structure of 3-manifolds, curvature flows, and moduli spaces of surfaces, providing insight into global geometric properties.
- **Complex Analysis:** Through the Weierstrass–Enneper representation, minimal surfaces correspond to holomorphic data, linking their geometry to analytic functions and conformal mappings.
- **Calculus of Variations:** They serve as canonical examples of critical points of the area functional, illustrating principles from variational methods and geometric measure theory.
- **Materials Science:** Minimal surface models describe interfaces in alloys, foams, and amphiphilic systems, where the structure tries to minimize surface energy.
- **Architecture:** Tensile membrane roofs, suspension bridges, and curved shell structures often use minimal surface geometries for strength and material efficiency.

- **Biology:** Cell membranes, soap films, and certain protein structures naturally adopt minimal surface configurations to ensure stability and minimize energy.

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