AN INTRODUCTION TO HYPERBOLIC GEOMETRY

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1. Abstract

This paper will serve as an introduction to hyperbolic geometry. We will begin with an explanation of the hyperbolic plane, its unique properties that differ from the Euclidean plane, and how to visualize hyperbolic geometry using the Poincaré disk model. We will discuss the fifth postulate of Euclidean geometry, using the Poincaré disk to see how this postulate does not apply to hyperbolic geometry. Next, we will examine how negative curvature affects the properties of triangles in the hyperbolic plane and derive a formula for calculating their area. Finally, we will examine how the hyperbolic structure of lettuce leaves increases their efficiency, which is one of many examples of hyperbolic geometry in nature.

2. Introduction

Definition 2.1 (Hyperbolic Geometry)**.** The curvature of hyperbolic geometry is constant and negative (as opposed to the constant zero curvature of Euclidean geometry). [6]

This causes the hyperbolic plane to act differently from the Euclidean plane in many aspects, including the following:

(1) The angle sum of a triangle is less than π.

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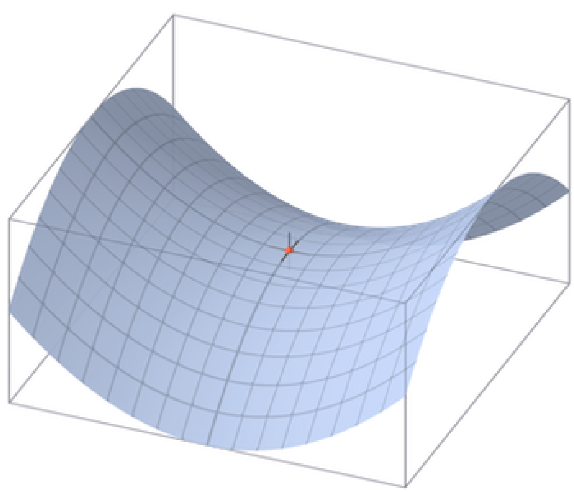

Figure 1. The Hyperbolic Plane

(2) All similar triangles are congruent.
(3) There are no lines that are everywhere equidistant from each other.
(4) If three angles of a quadrilateral are right angles, then the fourth angle must be less than a right angle.
(5) If a line intersects one of two parallel lines, it will not necessarily intersect the other.
(6) Lines that are parallel to the same line are not necessarily parallel to each other.
(7) Two intersecting lines may be parallel to the same line.
(8) Given two intersecting lines, there are infinitely many lines that do not intersect either of them.
(9) The circumference of a circle with radius r is greater than $2\pi r$.
(10) The hyperbolic plane can be tessellated in formations that do not work on the Euclidean plane.

[5] [7]

The hyperbolic plane curves away from itself in a saddle shape, as shown in Figure 1. Seeing how objects are stretched and curved more drastically as they approach the edges of the plane, we can visualize how the properties listed above work out in this warped space. The important thing to remember for this paper is that growth occurs exponentially as we approach infinity.

Euclidean geometry is famously built on the following five postulates:

(1) Any two points can be joined with a line segment.
(2) Any straight line segment can be extended indefinitely in a straight line.
(3) Given any line segment, a circle can be drawn with the segment as its radius and one endpoint as its center.
(4) All right angles are congruent.
(5) If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines must intersect each other on that side if extended far enough (Figure 2).

[3]

The fifth postulate breaks down when we try to apply it to hyperbolic geometry. This postulate is equivalent to the Parallel Postulate, which states that for a given line l and a point p not on that line, there is exactly one line including p that is parallel to l. [4]

Before we explain this, we will look at a common model for visualizing the hyperbolic plane called the Poincaré disk model.

3. The Poincaré Disk Model

The Poincaré disk is a common model of the hyperbolic plane. It represents the plane on a unit circle, with objects closer to the center appearing larger and objects closer to the edge appearing smaller. Since the hyperbolic plane is infinite, the disk model is an open set, and the points on the boundary of the unit circle are not included. Lines in the hyperbolic plane appear as arcs on the Poincaré disk since they stretch off towards infinity near the edges of the disk, even though they appear to run into the edge of the circle, as shown in Figure 3.

This model can demonstrate why the fifth postulate of Euclidean geometry does not hold up in the hyperbolic plane. Since lines in the hyperbolic plane can stretch in different directions as they approach infinity, it is possible for the interior angles

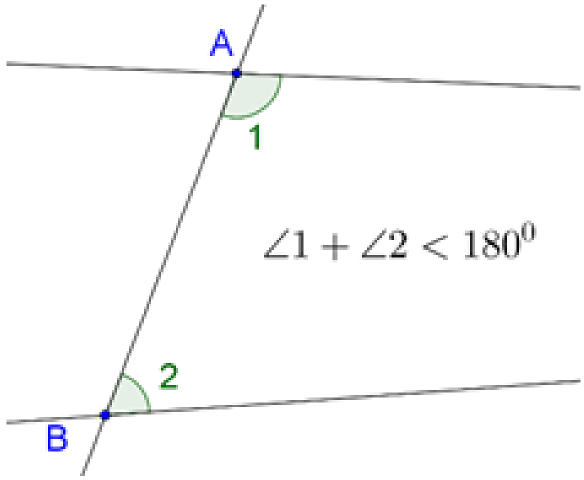

Figure 2. The two interior angles are narrower than two right angles

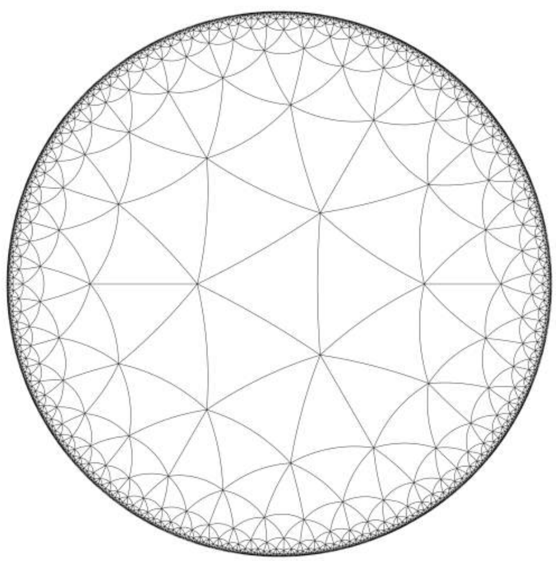

Figure 3. The Poincaré Disk Model

of two lines being intersected by a third to be less than two right angles but never intersect, or for multiple lines to intersect p while being parallel to l (Figure 4). [5]

4. Ideal Triangles

Definition 4.1 (Ideal Triangle)**.** An ideal triangle in the hyperbolic plane has its vertices stretched out towards infinity. All three angles are 0, and the area of the triangle is finite. [6]

It may seem odd that a triangle with vertices infinitely far apart would have a finite area, but each angle of the triangle becomes infinitely thin as it approaches

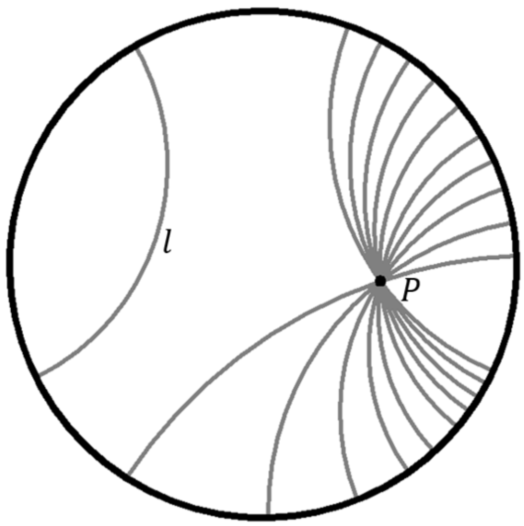

FIGURE 4. Many lines are parallel to l that include p

infinity so that the area between the sides grows infinitely small near the angle, and the total area converges to a finite value.

Referring back to the Poincaré disk from earlier, the ideal triangle has vertices at the edge of the circle, such as triangle DEF in Figure 5. We can see in the figure that the sides of the triangle appear as arcs, and the angles approach 0 as the arcs approach infinity (or, in this model, the edge of the circle).

Lemma 4.2. *All ideal triangles in the hyperbolic plane are congruent with an area of* π*.* [1]

Using this lemma, we can develop a formula to calculate the area of any triangle in the hyperbolic plane.

5. Area of Triangles

We will now find the area of a hyperbolic triangle ABC that does not have its vertices at infinity. To do so, we will consider an ideal triangle surrounding ABC. The excess area can be divided into three shims, which have two vertices at infinity and a third vertex supplementary to each angle of ABC, as shown in Figure 5.

Lemma 5.1 (Area of a Hyperbolic Shim)**.** *The area of a shim in the hyperbolic plane can be calculated using the formula* $S = \pi - \theta$ *where* S *is the area of the shim and* θ *is the non-ideal angle of the shim.* [6]

The total area of the ideal triangle, π, is equal to the area of the triangle ABC plus the areas of the shims. In Figure 5, the triangle has interior angles A, B, and C, and the shims have areas $\pi - (\pi - A)$, $\pi - (\pi - B)$, and $\pi - (\pi - C)$. Therefore, the area of the hyperbolic triangle plus A plus B plus C equals the area of the ideal triangle, π. Finally, we conclude that the area equals π minus the angle sum of the hyperbolic triangle.

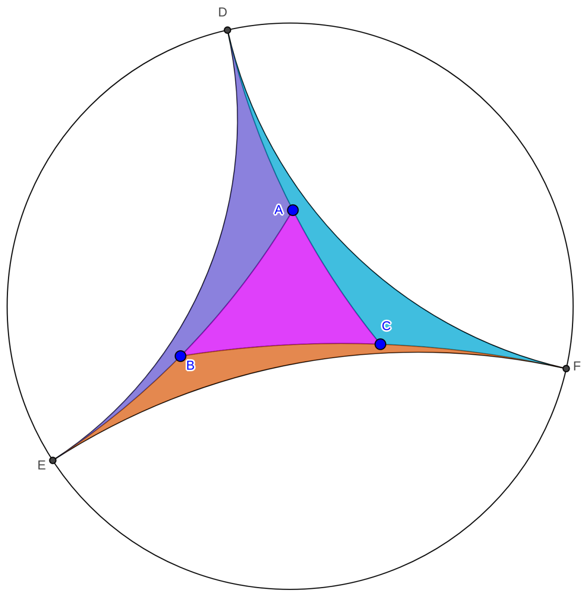

Figure 5. An Ideal Triangle

6. Math in Nature: Hyperbolic Lettuce

Hyperbolic geometry is commonly seen in nature to increase efficiency, such as influencing the shape of a lettuce leaf. This structure of lettuce allows it to maximize its surface area without taking up much space. Leaves that can increase their surface area for a given volume or thickness are more efficient because they are better able to absorb water and sunlight. The cells on the edges of lettuce leaves grow very quickly, whereas the cells in the center grow at a slower rate that is not sufficient for the leaf to lie flat. This causes it to take on a hyperbolic shape, being more compact in the center and taking up more space towards the edges, causing it to bunch up and wrinkle. [2]

7. Conclusion

The hyperbolic plane has constant negative curvature and does not comply with the fifth postulate of Euclidean geometry. Ideal triangles have an area of π, and non-ideal triangles can be calculated by subtracting their angle sum from π. Hyperbolic geometry is useful in the real world, and its structure allows lettuce leaves to increase their efficiency. Of course, it also has many more applications that are not covered in this paper.

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