Minimal surfaces

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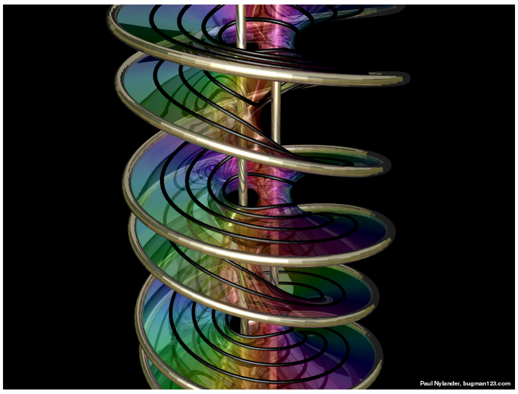
August 17, 2025

1 What are minimal surfaces?

Earlier in the course, we considered the problem of calculating the shortest distance between two points on a surface – i.e., a geodesic. Analogously, a minimal surface seeks to minimize its area across a boundary region being a fixed curve.

In 1849, Joseph Plateau demonstrated that soap films can be used as physical representations of some minimal surfaces, as the boundary is the curve formed by the wire frame; finding the existence of a minimal surface subject to a boundary is known as Plateau's problem. *Minimal surfaces* are surfaces that locally minimize their area, subject to some bounding constraint. A classical example commonly used to demonstrated the connection between minimal surfaces and soap films is the helicoid, swept out by a soap film on a helical frame.



Lagrange later expanded on this idea of minimal surfaces with the Euler-Lagrange (minimal-surface) equation for a graph $z = f(x, y)$:

$$\frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{1+|\nabla f|^2}}\right)+\frac{\partial}{\partial y}\left(\frac{f_y}{\sqrt{1+|\nabla f|^2}}\right)=0$$

Lagrange noted that a plane trivially satisfies this equation, and later, Euler and Meusnier discovered the catenoid and helicoid, respectively, which are now the most well known, classical minimal surfaces of the 18th century. Meusnier eventually established the connection between curvature and minimal surfaces, and derived what is now the canonical definition of minimal surfaces:

Definition 1 (Zero mean curvature)**.** A surface $S \subset \mathbb{R}^3$ is **minimal** if and only if it maintains zero mean curvature $H = 0$ at all points.

If we expand out the two partial derivative terms found above, we can express the equation solely in terms of partial derivatives of f:

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0$$

Recall from Exercise 6.1, however, that the mean curvature can be expressed with a numerator of the expression above. These definitions of zero mean curvature ($H = 0$) and the original equation Lagrange derived are thus equivalent.

2 An alternative definition for minimal surfaces

When we discussed geodesics, we analyzed a family of curves, all in close proximity of each other, and compared how the length of each curve in the family varied throughout the family. We then showed that the geodesic is a local minimum of a function that examines various different curves that passes through two points.

Definition 2 (Variational definition)**.** A surface $S \subset \mathbb{R}^3$ is **minimal** if and only if it is a critical point of the area functional for all compactly supported variations.

Likewise, we will analyze a family of surface patches $\sigma : U \to \mathbb{R}^3$, where τ varies across the open interval $-\epsilon < \tau < \epsilon$ for $\epsilon > 0$. Call the original surface σ^0. Choose a differential function $h : D \to \mathbb{R}$, where $D \subset U$. This normal variation φ, determined by h, is a map defined by

$$\varphi(u, v, \tau) = \sigma(u, v) + \tau h(u, v)N(u, v)$$

For a certain $\tau \in (-\epsilon, \epsilon)$, the map $\sigma^\tau : D \to \mathbb{R}^3$

$$\sigma^\tau(u, v) = \varphi(u, v, \tau)$$

is a parameterized surface that satisfies

$$\sigma_u^\tau = \sigma_u + \tau h N_u + \tau h_u N \sigma_v^\tau = \sigma_v + \tau h N_u + \tau h_u N$$

which follows from the product rule. After computing the first fundamental forms and plugging these constants into the formula for mean curvature, we find the area can be computed as

$$\mathcal{A}(\tau) = \int_D \sqrt{E^\tau G^\tau - (F^\tau)^2} d\mathcal{A}$$

$E^\tau G^\tau - (F^\tau)^2$ can be written as

$$E^\tau G^\tau - (F^\tau)^2 = (EG - F^2) - 2\tau h(LG - 2MF + NE) + O(\tau^2)$$

where $\lim_{O(\tau^2)\to 0} \frac{O(\tau^2)}{\tau} = 0$. If ϵ is sufficiently small, then σ^τ is a regular surface, and $\mathcal{A}(\tau)$ is a differentiable function. The area of $\mathcal{A}(\tau)$ is then

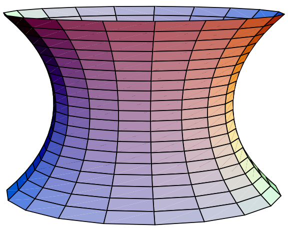
$$\mathcal{A}(\tau) = \int_D \sqrt{1 - 4\tau h H + O'(\tau^2)} \sqrt{EG - F^2}\, d\mathcal{A}$$

where $O'(\tau^2) = \frac{O(\tau^2)}{\sqrt{EG - F^2}}$. Then

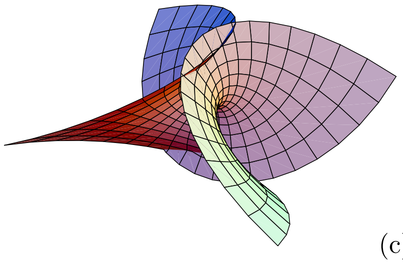
$$\mathcal{A}'(0) = -\int_D 2hH\sqrt{EG - F^2} d\mathcal{A}$$

which vanishes at $H = 0$ and thus has a local minimum at $\tau = 0$, as desired.

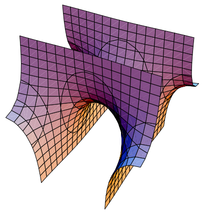
3 Examples of minimal surfaces



(a) Catenoid



(b) Enneper's surface



(c) Scherk's doubly periodic surface

Figure 1: Three classical minimal surfaces.

3.1 The catenoid

A catenoid is a surface obtained by revolving the curve $x = \frac{1}{a}\cosh(az)$ about the z-axis in the xz-plane. To simplify calculations, assume that $a = 1$. We can parameterize the catenoid as follows:

$$\sigma(u, v) = (\cosh(u)\cos(v), \cosh(u)\sin(v), u)$$

If we compute the first and second fundamental form constants, we find that $E = G = \cosh(u)^2$, $F = M = 0$, $L = -1$, and $N = 1$. Using the formula for mean curvature yields

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = 0$$

as desired.

3.2 Enneper's minimal surface

More complex is Enneper's minimal surface, which is parameterized by

$$\sigma(u, v) = \left(u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + vu^2, u^2 - v^2\right)$$

σ is technically not a surface patch, as it is not injective (self intersections can be seen in figure X), but we can partition σ's domain into various different open sets.

3.3 Scherk's doubly periodic minimal surface

Scherk's minimal surfaces are a pair of surfaces, the first of which we will be discussing:

$$\sigma(u, v) = (u, v, \log(\cos(u)) - \log(\cos(v)))$$

Scherk's minimal surface follows a checkerboard pattern of many separate unit cells that have vertices at points $(\frac{\pi}{2} + m\pi, \frac{\pi}{2} + n\pi), m, n \in \mathbb{Z}$. We will be analyzing the domain $U = \{(u, v) \in \mathbb{R}^2 : -\frac{\pi}{2} < u, v < \frac{\pi}{2}\}$. It can also easily be shown to be minimal by using Lagrange's equation from earlier:

$$\left(1 + \tan(v)^2\right)\left(-\sec^2(u)\right) - 2\left(-\tan(u)\right)\left(\tan(v)\right)(0) + \left(1 + \tan(u)^2\right)\left(\sec^2(v)\right)$$

which, by using the pythagorean trigonometric identity, is equal to zero.

4 A sprinkle of complex analysis and some other lemmas

Complex analysis is absolutely essential for constructing minimal surfaces, so we'll mention a few basic definitions necessary for Weierstrass-Enneper.

Definition 3. A map or parameterization is **isothermal** (or **conformal**) if all the angles on the surface are conserved after the transformation. It has the property of $E = G$ and $F = 0$.

Definition 4. A function $f : U \to \mathbb{R}$ is a **harmonic** function if f is at least of class C^2 and satisfies

$$f_{uu} + f_{vv} = 0$$

For instance, the function $f(u, v) = e^u \cos(v)$ is harmonic. We now have the tools to prove a fundamental identity necessary for Weierstrass:

Claim 5. Let σ be a regular and isothermal parameterized surface. Then

$$\sigma_{uu} + \sigma_{vv} = 2\lambda^2 HN$$

where HN is the mean curvature vector, and $\lambda^2 = E = G$.

Proof. Since we have $\sigma_u \cdot \sigma_u = \sigma_v \cdot \sigma_v$, we find that

$$\frac{d}{du}(\sigma_u \cdot \sigma_u) = \frac{d}{du}(\sigma_v \cdot \sigma_v)$$
$$\sigma_{uu} \cdot \sigma_u = \sigma_{vu} \cdot \sigma_v = 0$$

Since $\sigma_u \cdot \sigma_v = 0$, we have

$$\frac{d}{dv}(\sigma_u \cdot \sigma_v) = \sigma_{uv} \cdot \sigma_v + \sigma_u \cdot \sigma_{vv} = 0$$

and so $\sigma_{uu} \cdot \sigma_u = \sigma_{vu} \cdot \sigma_v = -\sigma_u \cdot \sigma_{vv}$. By the distributive properties of dot products, we thus have

$$(\sigma_{uu} + \sigma_{vv}) \cdot \sigma_u = 0$$

Similarly (by differentiating the original dot product with respect to v), we have that

$$(\sigma_{uu} + \sigma_{vv}) \cdot \sigma_v = 0$$

It then follows that $\sigma_{uu} + \sigma_{vv}$ is parallel to $\sigma_u \times \sigma_v$ and thus N.

Recall the definition for mean curvature in terms of first and second fundamental form constants:

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}$$

Since σ is isothermal $(F = 0, E = G)$, this can be reduced to

$$H = \frac{L + N}{2\lambda^2} = \frac{N \cdot (\sigma_{uu} + \sigma_{vv})}{2\lambda^2}.$$

So by multiplying out and taking the dot product of both sides with respect to the unit normal N, we have our result

$$\sigma_{uu} + \sigma_{vv} = 2\lambda^2 HN$$

as desired. □

A corollary of this is that if a surface $\sigma(u, v) = (x(u, v), y(u, v), z(u, v))$ is isothermal, it is is minimal if and only if each of its coordinate functions x, y, z are harmonic.

Surfaces can also be written in terms of complex coordinates, parameterized by a complex $\zeta = u + vi$, where $\zeta \in \mathbb{C}, (u, v) \in \mathbb{R}^2$. Before we start proving the fundamental lemma that we'll be using, we will establish some definitions from complex analysis first:

Definition 6. A function $f : U \subset \mathbb{C} \to \mathbb{C}$ is **analytic** (or **holomorphic**) if f can be written as

$$f(\zeta) = f_1(u, v) + if_2(u, v)$$

where $f_1, f_2 \in C^1$ and satisfy the Cauchy-Riemann equations

$$\frac{\partial f_1}{\partial u} = \frac{\partial f_2}{\partial v}, \frac{\partial f_1}{\partial v} = -\frac{\partial f_2}{\partial u}$$

Definition 7. Define φ to be

$$\varphi = \sigma_u - i\sigma_v$$

φ can be expressed as $\varphi = (\varphi_x, \varphi_y, \varphi_z)$; i.e., in terms of components.

Claim 8. σ is isothermal if and only if $\varphi_x^2 + \varphi_y^2 + \varphi_z^2 = 0$. If σ is isothermal, then σ is minimal if and only if φ is analytic.

Proof. Expanding out each term yields

$$\varphi_x^2 + \varphi_y^2 + \varphi_z^2 = E - G = 2iF$$

which is equal to zero if and only if $E - G = 0$ and $F = 0$.

Moreover, $\sigma_{uu} = -\sigma_{vv}$ (i.e., the condition for zero mean curvature from earlier) if

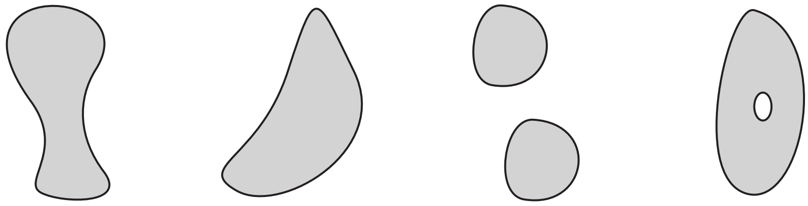
$$\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial u}\right) = -\frac{\partial}{\partial v}\left(\frac{\partial x}{\partial v}\right)$$

for each component function in σ, which then satisfy the negative case of the Cauchy-Riemann equations for φ. However, the positive case of the Cauchy-Riemann equations are trivially true in this case, so $\sigma_{uu} + \sigma_{vv} = 0$ if and only if φ is analytic. □

5 How can I construct a minimal surface?

Claim 8 is absolutely instrumental in our next claim on minimal surfaces; however, before we prove our next claim, we must define what it means for a surface to be **simply-connected**.

Definition 9. An open subset U of $\mathbb{R}^2$ is said to be **simply-connected** if every simple closed curve in U can be shrunk to a point staying inside U. Intuitively, this means that U has no "holes."



Claim 10. Let $\varphi = (\varphi_x, \varphi_y, \varphi_z)$, $\varphi : U \to \mathbb{C}^3$ – where U is simply connected – be an analytic vector-valued function that is nowhere zero and satisfies $\varphi_x^2 + \varphi_y^2 + \varphi_z^2 = 0$. Then there exists an isothermal parameterized minimal surface $\sigma : U \to \mathbb{R}^3$ with an associative analytic function φ. Furthermore, σ is uniquely determined by φ up to a translation.

Proof. Let $U \subset \mathbb{C}$ be simply connected and $\varphi = (\varphi_x, \varphi_y, \varphi_z) : U \to \mathbb{C}^3$ be analytic, nowhere zero, and satisfy $|\varphi|^2 = 0$.

Fix $z_0 \in U$ and set

$$F(z) := \int_{z_0}^{z} \varphi(\zeta)\, d\zeta \in \mathbb{C}^3, \qquad \sigma(z) := \Re F(z) \in \mathbb{R}^3.$$

Each component φ_k is analytic, so by Cauchy's theorem $\int_\gamma \varphi_k\, dz = 0$ for every closed loop γ in U. Since U is simply connected, F is path-independent and $F'(z) = \varphi(z)$; hence F is analytic.

With $\zeta = u + iv$ and using $F_{\bar{z}} = 0$,

$$F_u = \varphi, \qquad F_v = i\varphi,$$

so

$$\sigma_u = \Re\,\varphi, \qquad \sigma_v = \Re\,(i\varphi) = -\Im\,\varphi.$$

so $\varphi = \sigma_u - i\sigma_v$, as desired. From our conditions, we already know σ is isothermal, so we only need to prove that σ is unique up to a translation.

Say there is another isothermal minimal surface $\tilde{\sigma}$ that corresponds to the same analytic function σ. Then $\sigma_u = \tilde{\sigma}_u$ and $\sigma_v = \tilde{\sigma}_v$. Differentiating, we find that $\tilde{\sigma} - \sigma$ must be a constant. This must mean that $\tilde{\sigma}$ is a translation of σ obtained by translating by this constant, as desired. □

Example 11 (Helicoid)**.** Let's find the corresponding analytic function for a helicoid, which is an isothermal surface.

$$\begin{aligned}
\varphi(\zeta) &= \sigma_u - i\sigma_v \\
&= (-\sinh v \sin u - i \cosh v \cos u, \sinh v \cos u - i \cosh v \sin u, 1) \\
&= (-i\cos(u + vi), -i\sin(u + vi), 1) \\
&= (-i\cos(\zeta), -i\sin(\zeta), 1)
\end{aligned}$$

Note that φ satisfies the conditions in Claim 8. A complex integral of φ shows that σ is actually the conjugate surface (i.e. they have the same Weierstrass data rotated by a phase) of a catenoid.

Theorem 12 (Weierstrass-Enneper parameterization)**.** Let U be simply connected and $f, g : U \to \mathbb{C}$ be two analytic functions with $f \neq 0$ on all of U. Then $\varphi = (\varphi_x, \varphi_y, \varphi_z) : U \to \mathbb{R}^3$ expressed as

$$\begin{aligned}
\varphi_x &= \Re \int \frac{1}{2} f(\zeta) \left(1 - g(\zeta)^2\right) \, d\zeta \\
\varphi_y &= \Re \int \frac{i}{2} f(\zeta) \left(1 + g(\zeta)^2\right) \, d\zeta \\
\varphi_z &= \Re \int f(\zeta) g(\zeta) \, d\zeta
\end{aligned}$$

is an isothermal minimal surface.

Proof. The functions

$$\begin{aligned}
\psi_x(\zeta) &= \frac{1}{2} f(\zeta) \left(1 - g(\zeta)^2\right) \\
\psi_y(\zeta) &= \frac{i}{2} f(\zeta) \left(1 + g(\zeta)^2\right) \\
\psi_z(\zeta) &= f(\zeta) g(\zeta)
\end{aligned}$$

satisfy the equation $\psi_x^2 + \psi_y^2 + \psi_z^2 = 0$. Likewise,

$$\begin{aligned}
|\psi_x|^2 + |\psi_y|^2 + |\psi_z|^2 &= \frac{1}{4} \left|f(\zeta)^2\right| \left(\left|1 - g(\zeta)^2\right|^2 + \left|1 + g(\zeta)^2\right|^2 + 4 \left|g(\zeta)\right|^2 \right) \\
&= \frac{1}{2} \left|f(\zeta)\right|^2 \left(1 + |g(z)|^2\right)^2 \neq 0
\end{aligned}$$

Where we used the identity

$$\left|1 - \zeta^2\right|^2 + \left|1 + \zeta^2\right|^2 + 4 \left|\zeta\right|^2 = 2(1 + |\zeta|^2)^2$$

Since both critereon of Claim 10 are satisfied, φ is an isothermal minimal surface, as desired. ☐

Example 13 (Enneper's surface)**.** Let $f(\zeta) = 1$ and $g(\zeta) = \zeta$. Clearly, $f \neq 0$ and both functions are analytic. Furthermore, $\mathbb{C}$ is simply connected. Then the corresponding minimal surface φ is

$$\begin{aligned}
\varphi_x(u, v) &= \Re \int \frac{1}{2} \left(1 - \zeta^2\right) \, d\zeta = \Re \left(\frac{1}{2} \left(\zeta - \frac{1}{3} \zeta^3 \right) \right) \\
&= \frac{1}{2} \left(u - \frac{1}{3} u^3 + uv^2 \right) \\
\varphi_y(u, v) &= \Re \int \frac{i}{2} \left(1 + \zeta^2\right) \, d\zeta = \Re \left(\frac{i}{2} \left(\zeta + \frac{1}{3} \zeta^3 \right) \right) \\
&= \frac{1}{2} \left(-v + \frac{1}{3} v^3 - u^2 v \right) \\
\varphi_z(u, v) &= \Re \int \zeta \, d\zeta = \Re \left(\frac{1}{2} \zeta^2 \right) = \frac{1}{2} \left(u^2 - v^2 \right)
\end{aligned}$$

The surface $\varphi = \left(u + uv^2 - \frac{1}{3} u^3, -v + \frac{1}{3} v^3 - u^2 v, \frac{1}{2} u^2 - \frac{1}{2} v^2 \right)$ is called Enneper's surface, scaled by a factor of $\frac{1}{2}$. It can be shown that $\varphi(u, v)$ is isothermal.

References

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