THE BONNET-MYERS THEOREM

BRANDON MULIADI, SAHITI DOKE

1. Introduction

What would a finite, bounded universe look like? The intuitive answer is that a bounded universe would have some kind of boundary. This is not actually the case—our universe could be bounded yet still *complete*.

Definition 1.1. We say that a surface S is *complete* if any geodesic in S can be extended indefinitely in either direction.

A particle traveling in a straight line in such a bounded and complete universe could travel indefinitely without encountering any boundary, and could even return to its starting position. The Bonnet-Myers theorem allows us to show, given certain hypotheses, that a complete manifold actually must be compact (hence bounded). For this paper, we will prove a simpler version of the Bonnet-Myers Theorem, stated for surfaces:

Theorem 1.2 (Bonnet-Myers)**.** *Let S be a complete and connected surface with positive Gaussian curvature bounded away from zero. Then S is compact.*

Remark 1.3. If we weaken the conditions of the theorem to $K > 0$, then the conclusion does not hold. For a counterexample, consider the surface S $\{(x, y, f(x, y))\}$ where $f(x, y) = x^2 + y^2$. We can compute

$$\begin{aligned} K &= \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2} \\ &= \frac{2(2) - 0^2}{(1 + (2x)^2 + (2y)^2)^2} \\ &= \frac{4}{(1 + 4x^2 + 4y^2)^2} \\ &> 0. \end{aligned}$$

However, S is not bounded, ergo it is not compact.

We will prove compactness by bounding the *diameter* of S. In fact, the distance bound can be seen as a part of the Bonnet-Myers Theorem.

Definition 1.4. The *diameter* of a surface S is the supremum of the set of all distances between two points in S. (By distance, we mean the length of the shortest curve connecting the two points.)

Theorem 1.5 (Bonnet-Myers)**.** *Let S be a complete surface with $K \geq \delta > 0$. Then the diameter of S is at most $\frac{\pi}{\sqrt{\delta}}$.*

It is not entirely immediate that having a bound on the diameter of S implies that S is compact. We can show this using one version of the Hopf-Rinow theorem:

Theorem 1.6 (Hopf-Rinow)**.** *Let S be a connected smooth surface. Then the following statements are equivalent:*

(1) *The closed and bounded subsets of S are compact.*
(2) *S is a complete surface.*

If S is bounded, then S is a closed and bounded subset of S, which implies S is compact if S is also a complete surface by the Hopf-Rinow Theorem. So the second version of the Bonnet-Myers Theorem we've stated implies the first.

Remark 1.7. Where does the bound $\frac{\pi}{\sqrt{\delta}}$ come from? We imagine that a surface S achieving the maximum diameter would have a uniform Gauss curvature of δ—intuitively, the more curved a surface is, the faster it has to curve in on itself, so the curvature should be minimized everywhere. What does a surface with a uniform Gauss curvature of δ look like? One such surface is a sphere with radius $1/\sqrt{\delta}$. The diameter (in the differential geometry sense) of this sphere is $\pi/\sqrt{\delta}$, since the shortest curve between two antipodal points is half a great circle, which has length $\pi/\sqrt{\delta}$. It turns out that the sphere is the only surface that achieves the maximal diameter—this is a difficult result known as *Cheng's Maximal Diameter Theorem.*

Before going through the proof, we will first introduce some definitions.

2. Definitions

The approach presented in this paper is from [1].

Definition 2.1. Let $\alpha : [0, l] \to S$ be a regular parametrized curve with $s \in [0, l]$ as its parameter. A *variation* of α is a differentiable function $h : [0, l] \times (-\epsilon, \epsilon) \to S$ such that

$$h(s, 0) = \alpha(s)$$

for all $s \in [0, l]$.

Definition 2.2. With notation as before, we say that h is *proper* if

$$h(0, t) = \alpha(0),$$
$$h(l, t) = \alpha(l),$$

for $t \in (-\epsilon, \epsilon)$.

In other words, we say that h is a proper variation if h does not "vary" the endpoints of α.

Definition 2.3. The *variational vector field* of h is defined as

$$V(s) = \frac{\partial h}{\partial t}(s, 0)$$

for $s \in [0, l]$. Note that if h is proper, then

$$V(0) = V(l) = 0.$$

Definition 2.4. Let $\mathbf{p}$ be a point on a surface S, and let $\mathbf{v} \in T_{\mathbf{p}}S$. If $\mathbf{v} \neq 0$, then let γ be the unique geodesic with $\gamma(0) = \mathbf{p}$ and $\dot{\gamma}(0) = \mathbf{v}$. Then we define

$$\exp_{\mathbf{p}}(\mathbf{v}) = \gamma(1).$$

If $\mathbf{v} = 0$, then we define $\exp_{\mathbf{p}}(\mathbf{v}) = \mathbf{p}$. We call exp the *exponential map.*

We may think of $\exp_{\mathbf{p}} \mathbf{v}$ as following the unique geodesic with derivative $\mathbf{v}$ at $\mathbf{p}$ for time 1, laying down a distance of $|\mathbf{v}|$.

Proposition 2.5. *If V is a tangent vector field to a curve α, then there exists a variation $h : [0, l] \times (-\epsilon, \epsilon)$ of α such that V is the variational vector field of h. Furthermore, if $V(0) = V(l) = 0$ then we may choose h to be proper.*

Proof. We first choose a $\delta > 0$ such that $|v| < \delta$ implies $\exp_{\alpha(s)} v$ is well defined for every $s \in [0, l]$. We do not prove this key step, which requires another lemma; a proof can be found in [1]. Now, let M be an upper bound on $|V(s)|$ and let $\epsilon = \delta/M$. Define $h(s,t) = \exp_{\alpha(s)} tV(s)$. It can be checked that $h(s, 0) = \alpha(s)$ and $\frac{\partial h}{\partial t}(s, 0) = V(s)$. If $V(0) = V(l) = 0$, then we can easily see that $h(0,t) = h(0,0) = \alpha(0)$ and $h(l,t) = h(l,0) = \alpha(l)$ since $\frac{\partial h}{\partial t} = 0$ for $s = 0$ and $s = l$. ■

Definition 2.6. We write $\frac{\mathrm{D}}{\partial x}$ to denote taking covariant derivatives with respect to x.

3. Variations of Arclength

We now require tools to investigate how the arclength of $\alpha(s)$ changes as we move to variations on $h(s,t)$. In what follows, we use the notation of S, V, h, and so on as before, and assume that h is proper. Now define $L : (-\epsilon, \epsilon) \to \mathbb{R}$ as

$$L(t) = \int_0^l \left| \frac{\partial h}{\partial s}(s,t) \right| ds.$$

We would like a formula for $L''(0)$, so that we understand the behavior of L at 0. We now prove a series of lemmas.

Lemma 3.1. *There exists a $\delta > 0$ such that $L(t)$ is differentiable for $t \in (-\delta, \delta)$, and the derivative is given by*

$$L'(t) = \int_0^l \frac{\partial}{\partial t} \left| \frac{\partial h}{\partial s}(s,t) \right| ds$$

(that is, we just differentiate under the integral sign.)

Proof. Note that $\left| \frac{\partial h}{\partial s}(s, 0) \right| = 1$ since α is parametrized by arc length. Now, since $[0, l]$ is compact, there exists a $\delta > 0$ with $\delta \leq \epsilon$ such that

$$|t| \leq \delta \Rightarrow \left| \frac{\partial h}{\partial s}(s,t) \right| \neq 0$$

for $s \in [0, l]$. The absolute value of a nonzero differentiable function is differentiable, so $\left| \frac{\partial h}{\partial s}(s,t) \right|$ is differentiable for $t \in (-\delta, \delta)$. Then by the Leibniz integral rule, we may differentiate under the integral sign:

$$L'(t) = \int_0^l \frac{\partial}{\partial t} \left| \frac{\partial h}{\partial s}(s,t) \right| ds.$$

■

The following lemmas below will help us compute $L''(0)$.

Lemma 3.2. *Let $w(t)$ be a differential vector field along α. Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Then*

$$\frac{D}{\partial t} f(t)w(t) = f(t)\frac{D}{\partial t}w + \frac{df}{dt}w(t)$$

Proof. Let $(\quad)_T$ denote the tangential component of $(\quad)$.

$$\begin{aligned} \frac{\mathrm{D}}{\partial t} f(t)w(t) &= \left(\frac{df}{dt}w + f\frac{dw}{dt}\right)_T \\ &= \frac{df}{dt}w + f\frac{\mathrm{D}}{\partial t}w. \end{aligned}$$

∎

Lemma 3.3. *Let $v(t)$ and $w(t)$ be differentiable vector fields along α. Then*

$$\frac{d}{dt}\langle v(t), w(t)\rangle = \left\langle \frac{D}{\partial t}v, w \right\rangle + \left\langle v, \frac{D}{\partial t}w \right\rangle$$

Proof. Let $(\quad)_T$ and $(\quad)_N$ denote the tangential and normal components of $(\quad)$, respectively. First of all, we have

$$\frac{d}{dt}\langle v, w\rangle = \left\langle \frac{dv}{dt}, w \right\rangle + \left\langle v, \frac{dw}{dt} \right\rangle.$$

Note that

$$\left\langle \frac{dv}{dt}, w \right\rangle = \left\langle \left(\frac{dv}{dt}\right)_T, w \right\rangle + \left\langle \left(\frac{dv}{dt}\right)_N, w \right\rangle = \left\langle \frac{\mathrm{D}}{\partial t}v, w \right\rangle$$

where $\left\langle \left(\frac{dv}{dt}\right)_N, w \right\rangle = 0$ since $\left(\frac{dv}{dt}\right)_N$ is normal to S and w is tangent to S by definition. Similarly $\left\langle v, \frac{dw}{dt} \right\rangle = \left\langle v, \frac{\mathrm{D}}{\partial t}w \right\rangle$. Thus

$$\frac{d}{dt}\langle v, w\rangle = \left\langle \frac{\mathrm{D}}{\partial t}v, w \right\rangle + \left\langle v, \frac{\mathrm{D}}{\partial t}w \right\rangle.$$

∎

Lemma 3.4. *Let $h : [0,l] \times (-\epsilon, \epsilon) \subset \mathbb{R}^2 \to S$ be a differentiable mapping. Then*

$$\frac{D}{ds}\frac{\partial h}{\partial t}(s,t) = \frac{D}{dt}\frac{\partial h}{\partial s}(s,t).$$

Proof. Let $\sigma(u,v) : U \to S$ be a a surface patch of S containing $h(s,t)$, and suppose that $h(s,t) = \sigma(h_1(s,t), h_2(s,t))$ in this patch. At $s = s_0$, we know that $\frac{\partial h}{\partial s}(s_0,t_0)$ is tangent to the curve $h(s,t_0)$, so

$$\frac{\partial h}{\partial s}(s_0,t_0) = \frac{\partial h_1}{\partial s}(s_0,t_0)\sigma_u + \frac{\partial h_2}{\partial s}(s_0,t_0)\sigma_v.$$

Since our choice of (s_0,t_0) was arbitrary, we get that

$$\frac{\partial h}{\partial s} = \frac{\partial h_1}{\partial s}\sigma_u + \frac{\partial h_2}{\partial s}\sigma_v.$$

Similarly,

$$\frac{\partial h}{\partial t} = \frac{\partial h_1}{\partial t}\sigma_u + \frac{\partial h_2}{\partial t}\sigma_v.$$

We can now use the formula for the covariant derivative in terms of the Christoffel symbols and σ_u, σ_v to verify that both sides have the same coefficients of σ_u and σ_v, so they are equal. ■

Now, we derive a formula for $L'(t)$ and show that $L'(0) = 0$.

Proposition 3.5. *For* $t \in (-\delta, \delta)$,

$$L'(0) = -\int_0^l \langle A(s), V(s)\rangle \, ds$$

where $A(s) = \frac{D}{\partial s}\frac{\partial h}{\partial s}(s, 0)$.

Proof. We first rewrite $L(t) = \int_0^l \left|\frac{\partial h}{\partial s}\right| ds$ as

$$L'(t) = \int_0^l \frac{d}{dt}\left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s}\right\rangle^{1/2} ds.$$

Applying Lemma 3.3 and Lemma 3.4 to compute the integrand, we get

$$\begin{aligned} L'(t) &= \int_0^l \frac{1}{2}\left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s}\right\rangle^{-1/2} \cdot 2\left\langle \frac{\mathrm{D}}{\partial t}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial s}\right\rangle ds \\ &= \int_0^l \frac{\left\langle \frac{\mathrm{D}}{\partial t}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial s}\right\rangle}{\left|\frac{\partial h}{\partial s}\right|} ds \\ &= \int_0^l \frac{\left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle}{\left|\frac{\partial h}{\partial s}\right|} ds. \end{aligned}$$

We know that $\left|\frac{\partial h}{\partial s}(s, 0)\right| = 1$ since $h(s, 0) = \alpha(s)$ is parametrized by arc-length, so this simplifies to

$$L'(0) = \int_0^l \left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle ds.$$

From Lemma 3.3, we know that

$$\frac{\partial}{\partial s}\left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle = \left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle + \left\langle \frac{\partial h}{\partial s}, \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}\right\rangle$$

so we have

$$L'(0) = \int_0^l \frac{\partial}{\partial s}\left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle ds - \int_0^l \left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle ds.$$

The first term on the RHS is 0 since h is proper and thus $\frac{\partial h}{\partial t}(0, 0) = \frac{\partial h}{\partial t}(l, 0) = 0$. We conclude that

$$\begin{aligned} L'(0) &= -\int_0^l \left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle ds \\ &= -\int_0^l \langle A(s), V(s)\rangle \, ds. \end{aligned}$$

■

We can use the above proposition to prove the interesting fact that α is a geodesic iff $L'(0) = 0$.

Proposition 3.6. *A regular parametrized curve $\alpha(s)$ parametrized by arc-length is a geodesic iff for every proper variation h, $L'(0) = 0$.*

Proof. The "only if" direction of the proof is simple. We know that the acceleration vector, $\frac{\mathrm{D}}{\partial s}\frac{\partial \alpha}{\partial s}$, of a geodesic is zero since geodesics are constant speed. Hence, $L'(0) = 0$.

We now prove the other direction. Suppose $L'(0) = 0$ for every h. Define a vector field $V(s) = f(s)A(s)$ where $f : [0, l] \to \mathbb{R}$ is a real differentiable function with $f(0) = f(l) = 0$ and $f(s) > 0$ for $s \in (0, l)$. Let h be a proper variation with variational vector field $V(s)$. Then we compute

$$\begin{aligned} L'(0) &= -\int_0^l \langle f(s)A(s), A(s)\rangle \, ds \\ &= -\int_0^l f(s)|A(s)|^2 \, ds \\ &= 0. \end{aligned}$$

Since f is nonnegative, we have $f(s)|A(s)|^2 \geq 0$. Thus

$$f(s)|A(s)|^2 = 0$$

identically. This shows that $A(s)$ must be 0. If $|A(s_0)| \neq 0$ for some s_0, then in particular $|A(s_1)| \neq 0$ for each $s_1 \in (s_0 - \epsilon, s_0 + \epsilon)$ for some $\epsilon \in 0$. Choosing $s_1 \in (0, l)$, we have $f(s_1) \neq 0$, so now $f(s_0)|A(s_0)|^2 \neq 0$, contradiction. Ergo, $A(s) = 0$ when $s \in (0, l)$ and by continuity, we get that $A(0) = A(l) = 0$. Since $A(s)$ is identically zero, α is a geodesic. ∎

From now on, we only consider proper, orthogonal variations of geodesics $\gamma : [0, l] \to S$ to make our calculations easier. (An orthogonal variation satisfies $\langle V(s), \gamma'(s)\rangle = 0$.) We require two more lemmas before for the computation of $L''(0)$. The proofs to these lemmas are omitted, as they are quite computational, but the interested reader can find proofs in [1].

Lemma 3.7. *Let $\mathbf{x}(u, v) : U \to S$ be a parametrization at point $\mathbf{p} \in S$ of a regular surface S and let K be the Gaussian curvature. Then,*

$$\frac{D}{\partial v}\frac{D}{\partial u}\mathbf{x_u} - \frac{D}{\partial u}\frac{D}{\partial v}\mathbf{x_u} = K(\mathbf{x_u} \wedge \mathbf{x_v}) \wedge \mathbf{x_u}.$$

The preceding lemma is required to prove the next one.

Lemma 3.8.

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = K(s, t)\left(\frac{\partial h}{\partial s} \wedge \frac{\partial h}{\partial t}\right) \wedge V$$

where $K(s, t)$ is the curvature at point $h(s, t)$.

We now have all the lemmas needed to make lemmanade (and write a formula for $L''(0)$).

Proposition 3.9. *For $t \in (-\delta, \delta)$ we have*

$$L''(0) = \int_0^l \left|\frac{D}{\partial s}V(s)\right|^2 - K(s)|V(s)|^2 \, ds.$$

Proof. From Proposition 3.5, we have

$$L'(t) = \int_0^l \frac{\left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle}{\left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s}\right\rangle^{1/2}}\, ds$$

for $t \in (-\delta, \delta)$, as defined earlier. Differentiating gives

$$L''(t) = \int_0^l \frac{\frac{d}{dt}\left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle \left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s}\right\rangle^{1/2}}{\left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s}\right\rangle} - \int_0^l \frac{\left(\left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle\right)^2}{\left|\frac{\partial h}{\partial s}\right|^{3/2}}.$$

For $t = 0$, we have $|\frac{\partial h}{\partial s}(s,0)| = 1$. Furthermore,

$$\frac{d}{ds}\left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle = \left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle + \left\langle \frac{\partial h}{\partial s}, \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}\right\rangle.$$

Since γ is a geodesic, the acceleration vector $(\frac{D}{\partial s}\frac{\partial h}{\partial s})$ is 0, and since we're only dealing with orthogonal variations, $\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s}\rangle = 0$ at $t = 0$. So, we can write $L''(0)$ as

$$L''(0) = \int_0^l \frac{d}{dt}\left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle (s,0)\, ds.$$

We rewrite the integrand for convenience

$$\begin{aligned}\frac{d}{dt}\left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle &= \left\langle \frac{\mathrm{D}}{\partial t}\frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle + \left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\mathrm{D}}{\partial t}\frac{\partial h}{\partial s}\right\rangle \\ &= \left\langle \frac{\mathrm{D}}{\partial t}\frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle - \left\langle \frac{\mathrm{D}}{\partial s}\frac{\mathrm{D}}{\partial t}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle + \left\langle \frac{\mathrm{D}}{\partial s}\frac{\mathrm{D}}{\partial t}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle + \left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\mathrm{D}}{\partial t}\frac{\partial h}{\partial s}\right\rangle.\end{aligned}$$

We know (using Lemma 3.3) that for $t = 0$,

$$\frac{d}{dt}\left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle = \left\langle \frac{\mathrm{D}}{\partial s}\frac{\mathrm{D}}{\partial t}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle + \left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial s}\right\rangle = \left\langle \frac{\mathrm{D}}{\partial s}\frac{\mathrm{D}}{\partial t}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle,$$

noting that $\frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial s} = 0$ since $h(s,0) = \gamma(s)$ is a geodesic. We use Lemma 3.10 and the properties of orthogonal variation to further simplify our integrand:

$$\begin{aligned}\left\langle \frac{\mathrm{D}}{\partial t}\frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle - \left\langle \frac{\mathrm{D}}{\partial s}\frac{\mathrm{D}}{\partial t}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle &= K(s)\left\langle \left(\frac{\partial h}{\partial s} \wedge \frac{\partial h}{\partial t}\right) \wedge \frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}\right\rangle \\ &= -K\left\langle |V(s)|^2 \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s}\right\rangle \\ &= -K|V(s)|^2.\end{aligned}$$

So, we rewrite the integrand to get

$$
\begin{aligned}
L''(0) &= -\int_0^l K|V(s)|^2 + \left\langle \frac{\mathrm{D}}{\partial s}\frac{\partial h}{\partial t}, \frac{\mathrm{D}}{\partial t}\frac{\partial h}{\partial s} \right\rangle ds \\
&= \int_0^l -K|V(s)|^2 + \left| \frac{\mathrm{D}}{\partial s} V(s) \right|^2 ds \\
&= \int_0^l \left| \frac{\mathrm{D}}{\partial s} V(s) \right|^2 - K|V(s)|^2 \, ds
\end{aligned}
$$

and we're done. ∎

There is one final ingredient we need for the proof, the *Hopf-Rinow Theorem* (in a different form than what we used earlier).

Theorem 3.10 (Hopf-Rinow Theorem). *Let S be a complete surface. For any two points $\mathbf{p}, \mathbf{q} \in S$, there exists a geodesic of minimal length connecting $\mathbf{p}$ and $\mathbf{q}$.*

4. The Bonnet-Myers Theorem

We now have enough to prove the Bonnet-Myers Theorem. Given a geodesic γ of minimal length connecting two points of S, we will choose a suitable $V(s)$, compute $L''(0)$, and show that $L''(0) \geq 0$ implies the desired bound on the length of γ.

Theorem 4.1 (Bonnet-Myers). *Let S be a complete surface with $K \geq \delta > 0$. Then S is compact. In particular, the diameter of S is at most $\frac{\pi}{\sqrt{\delta}}$.*

Proof. Let $\mathbf{p}, \mathbf{q} \in S$. By the Hopf-Rinow Theorem, there exists a geodesic γ of minimal length connecting $\mathbf{p}$ and $\mathbf{q}$. Let l be the length of γ, and suppose that γ is arclength parametrized, so that γ is a function from $[0, 1]$ to S. We now construct a variation of γ. Let $\mathbf{w}(0) \in T_{\gamma(0)}S$ be a unit vector with $\mathbf{w}(0) \cdot \dot{\gamma}(0) = 0$. For $s \in (0, l]$, let $w(s)$ be the parallel transport of $w(0)$ along γ to $\gamma(s)$. We note now that $|w(s)| = 1$ and $\mathbf{w}(s) \cdot \dot{\gamma}(s) = 0$ for all $s \in [0, l]$. Now define the variational vector field

$$
V(s) = \sin\left(\frac{\pi}{l}s\right) w(s)
$$

and note that we can choose a proper variation h of γ with variational vector field $V(s)$. With this h, we compute

$$
\begin{aligned}
L''(0) &= \int_0^l \left| \frac{\mathrm{D}}{\partial s} V(s) \right|^2 - K(s)|V(s)|^2 \, ds \\
&= \int_0^l \left(\frac{\pi}{l} \cos\left(\frac{\pi}{l}s\right) \right)^2 - K(s) \sin^2\left(\frac{\pi}{l}s\right) \, ds \\
&= \int_0^l \frac{\pi^2}{l^2} \cos^2\left(\frac{\pi}{l}s\right) - K(s) \sin^2\left(\frac{\pi}{l}s\right) \, ds.
\end{aligned}
$$

Here we computed that $\frac{\mathrm{D}}{\partial s} V(s) = \left(\frac{\mathrm{D}}{\partial s} \sin\left(\frac{\pi}{l}s\right)\right) w(s) + \sin\left(\frac{\pi}{l}s\right) \left(\frac{\mathrm{D}}{\partial s} w(s)\right) = \frac{\pi}{l} \cos\left(\frac{\pi}{l}s\right) w(s)$, since $\frac{D}{\partial s} w(s) = 0$ by the definition of parallel transport. Now note that if we replace K with $\frac{\pi^2}{l^2}$, this integral just becomes 0, since $\int_0^l \cos^2\left(\frac{\pi}{l}s\right) - \sin^2\left(\frac{\pi}{l}s\right) \, ds = 0$. We use this trick to simplify:

$$\begin{aligned} L''(0) &= \int_0^l \left(\frac{\pi^2}{l^2} \cos^2\left(\frac{\pi}{l}s\right) - \frac{\pi^2}{l^2}\sin^2\left(\frac{\pi}{l}s\right)\right) ds + \int_0^l \left(\frac{\pi^2}{l^2} - K(s)\right) \sin^2\left(\frac{\pi}{l}s\right) ds \\ &= \frac{\pi^2}{l^2}\int_0^l \cos\left(\frac{2\pi}{l}s\right) ds + \int_0^l \left(\frac{\pi^2}{l^2} - K(s)\right)\sin^2\left(\frac{\pi}{l}s\right) ds \\ &= \int_0^l \left(\frac{\pi^2}{l^2} - K(s)\right)\sin^2\left(\frac{\pi}{l}s\right) ds. \end{aligned}$$

Since γ is length-minimizing, $L''(0) \geq 0$. Thus $\frac{\pi^2}{l^2} - \delta \geq \frac{\pi^2}{l^2} - K(s) \geq 0$, so $l \leq \frac{\pi}{\sqrt{\delta}}$ as desired. ■

References

[1] Manfredo P. do Carmo. *Differential geometry of curves and surfaces*. Prentice-Hall, 1987.