The Tait-Kneser Theorem

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1 Introduction

The Tait-Kneser theorem describes the arrangement of osculating circles along a plane curve. It also provides insight into the geometric properties of curves.

It was originally demonstrated by Peter Guthrie Tait in the late 19th century and then was elaborated on by Adolf Kneser in the early 20th century.

This paper will explain the theorem, its proof and significance as well as include some examples and connected theorems.

2 Background Information

For every point on a smooth (meaning continuously differentiable) plane curve with non-vanishing curvature there can be an osculating circle which is tangent to the curve at the point in question, and has the same curvature. See Figure 1.

We can approximately find the osculating circle by picking a point $\mathbf{p}$ on a curve γ and picking two other points near the point $\mathbf{p}$ and drawing the circle through these 3 points. The curvature is also $1/R$ where R is the radius of the osculating circle.

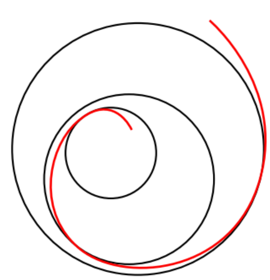

Figure 1: Osculating circles which are nested

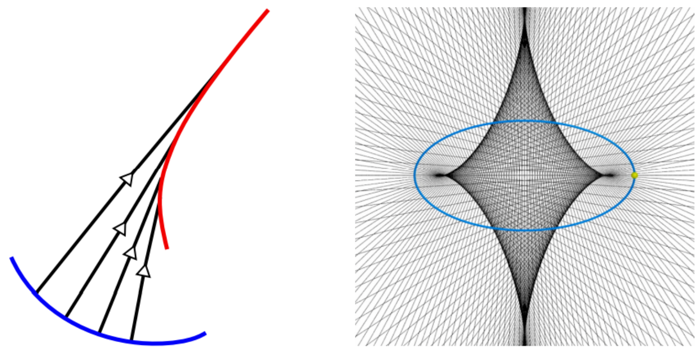

Figure 2: Left: the red curve is the evolute of the blue curve; Right: the evolute of an ellipse

Definition 1. A monotonic curve is a curve where the curvature constantly increases or decreases.

Definition 2. Nested circles mean that one circles interior is contained within another circles interior (as seen in Figure 1).

Definition 3. The curves are disjoint if and only if they do not overlap at any point

Definition 4. The evolute of the curve is a curve that describes the locus of centers of osculating circles. See Figure 2.

3 Tait-Kneser Theorem

Theorem 1(Tait-Kneser). *The osculating circles of a non-vanishing plane curve with monotonic positive curvature are pairwise disjoint and nested.*

[4] [3] Tait's proof is quite simple and can be quoted almost verbatim:

For, if A, B, be any two points of the evolute, the chord AB is the distance between the centers of two of the circles, and is necessarily less than the arc AB, the difference of their radii. Thus one of the circles lies wholly within the other.

When the curve has points of maximum or minimum curvature, there are corresponding cusps on the evolute ; and pairs of circles of curvature, whose centers lie on opposite sides of the cusp, C, may intersect because the, chord AB may now exceed the difference between CA and CB. Recall the cusp is the point where two branches of a curve meet.

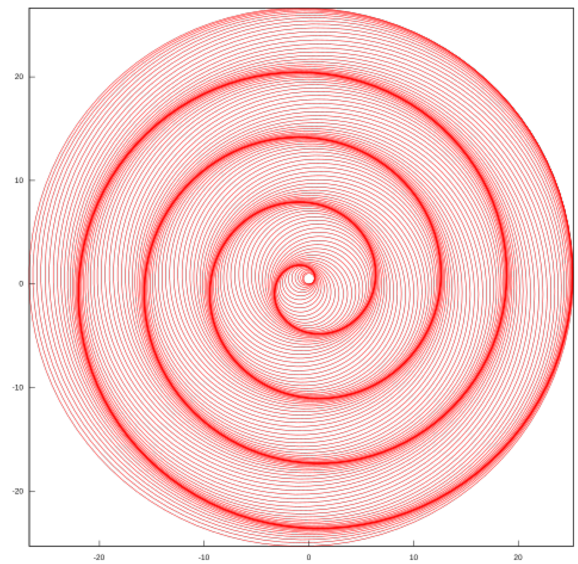

Figure 3: Archimedean spiral with osculating circles

There are variations of proofs of the Tait-Kneser theorem, for example, the one using Lorentzian geometry as seen in [1]

Example 1. The Archimedean spiral (as seen in Figure 3) is given by the formula $r = a\theta + b$, where r is the radius from the center and a and b are constants.

For the Tait-Kneser theorem to be applied the curvature must be non-vanishing and monotonic.

The curvature, $\kappa(t)$, can be derived through the formula $\frac{f'g''-f''g'}{((f')^2-(g')^2)^{\frac{3}{2}}}$.

Using the parameterization of the Archimedean spiral, $(rt\cos(t), rt\sin(t))$, we get:

$\kappa(t) = \frac{2+t^2}{(r(1+t^2))^{\frac{3}{2}}}$

We can see that for any $t > 0$, $\kappa(t) \neq 0$, thus the curve is non-vanishing. To show the curvature is monotonic, we find

$\kappa'(t) = \frac{-t(t^2+4)}{r(1+t^2)^{\frac{5}{2}}}$

As we can see that $\kappa'(t) \neq 0$ for any $t > 0$, it means the curvature is constantly increasing or decreasing with no critical points, thus meaning the curvature is monotonic.

As the curvature follows both conditions of being smooth and monotonic the Tait-Kneser theorem can be applied. This is also true for the logarithmic spiral.

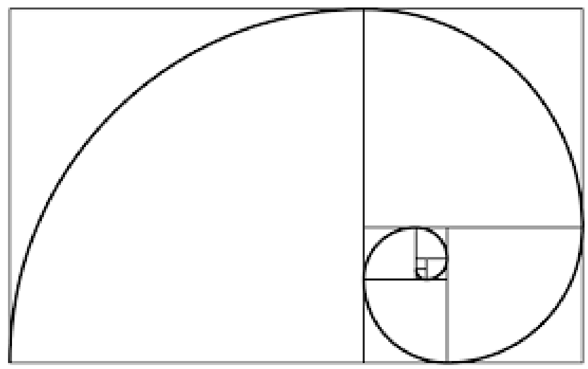

Figure 4: Fibonacci Spiral

Example 2. Although the Fibonacci spiral (as seen in Figure 4) at first glance may look like an example of the Tait-Kneser theorem, it is not. To prove that the spiral is not an example of the Tait-Kneser theorem, we can use the same methodology as in *Example 1* to find the curvature, where in this case the results would show that the curvature is disjoint, changing after each quarter-circle segment. The discontinuity means that the curve is not smooth, thus contradicting the conditions for the Tait-Kneser theorem to apply.

4 Related Concepts

Recall that a vertex is a local extrema point of $\kappa(t)$

Theorem 2(Four-Vertex theorem). *A simple, smooth, closed plane curve has at least 4 vertices.*

Interestingly, both the Four-Vertex theorem and the Tait-Kneser theorem were expanded upon by Adolf Kneser. Additionally, both theorems concern the properties of osculating circles along plane curves.

Example 3. Kepler Orbits are plane conics (ellipses, hyperbolas or parabolas). Certain Kepler Orbits such as ellipses follow both the Four-Vertex theorem and the Tait-Kneser theorem. However, the Tait-Kneser theorem can only apply locally when the local neighborhood the theorem is being applied has no extrema points. This is because at extrema points the curvature is 0 and therefore not increasing or decreasing making the curvature not monotonic and preventing the theorem from being applied (as seen in Figure 5). On the other hand the Four-Vertex theorem applies globally as it refers to the entire closed, plane curve and does not have restrictions such as the monotonicity of the curve.

Definition 5. Space curves are curves that exist in a 3-dimensional space. [2]

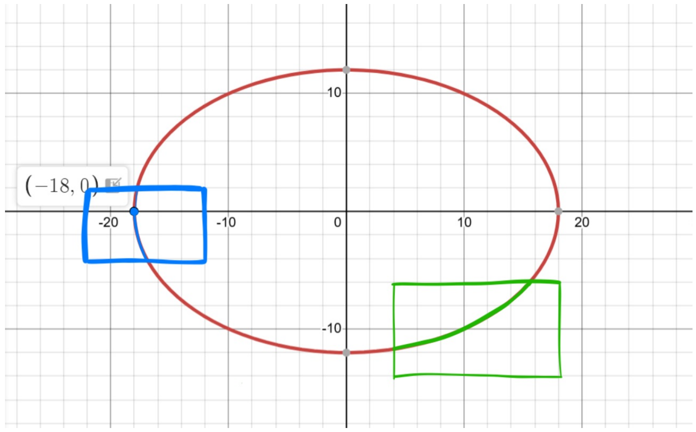

Figure 5: Blue: An example of where the Tait-Kneser theorem would not apply; Green: An example of where the Tait-Kneser theorem would apply

Although the Tait-Kneser is about plane curves, in some cases it can exist for space curves. However, instead of osculating circles, there are osculating spheres.

The Tait-Kneser theorem has also been adapted to be osculating conics rather than osculating circles.

References

[1] Gil Bor, Connor Jackman, and Serge Tabachnikov. Variations on the tait-kneser theorem, 2021.

[2] Dmitry Fuchs, Ivan Izmestiev, Matteo Raffaelli, Gudrun Szewieczek, and Serge Tabachnikov. Differential geometry of space curves: Forgotten chapters. *The Mathematical Intelligencer*, 46(1):9–21, July 2023.

[3] E. Ghys, S. Tabachnikov, and V. Timorin. Osculating curves: around the tait-kneser theorem, 2012.

[4] Tait. Note on the circles of curvature of a plane curve. *Proceedings of the Edinburgh Mathematical Society*, 14:26–26, 1895.