Symplectic Geometry

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1 Introduction

Symplectic geometry is a branch of differential geometry and topology that studies smooth manifolds equipped with a closed, non-degenerate 2-form. Two centuries ago, the field provided mathematical foundations to classical mechanics, and in particular Hamiltonian mechanics; a classical mechanical system can be modeled by the phase space which is a symplectic space. Hence, many mathematicians consider the symplectic structure's date of origin to be in 1811 when Joseph-Louis Lagrange wrote the work *Analytical Mechanics*.

The origins for the field are still very debateable. While some argue that symplectic manifolds arose from classical mechanics, this claim is not entirely correct as the first manifolds endowed with a symplectic structure were the Kähler manifolds defined by Erich Kähler in 1933. Even so, symplectic manifolds were not explicitly defined until 1950 by Charles Ehresmann who wanted to find out whether any $2n$ real dimensional manifolds had a complex structure. Next, "symplectic geometry" was used for the first time in 1943 by Carl Ludwig Siegel; however, he referred to symplectic geometry as a generalization of hyperbolic geometry to $\frac{1}{2}n(n+1)$ complex dimensions, and although his half-space is indeed a symplectic space, nowadays symplectic geometry refers to a much broader range of concepts. Another founding pioneer of the field was Jean-Marie Souriau, who in 1953 gave a presentation titled "Géométrie symplectique différentielle."

Despite its ties with mechanics, this branch of mathematics is considered an independent, flourishing field: the geometry of symplectic manifolds. In this expository paper, we present a self-contained introduction to the field of symplectic geometry. We define its key objects and present some significant theorems with their proofs.

2 Symplectic Manifolds

2.1 Smooth Manifolds

First, we will define some key foundational objects. Let X be a set. A *topology* on X is a collection $\mathcal{T}$ of subsets of X, called open subsets, satisfying

1. X and $\emptyset$ are open.

2. The union of any family of open subsets is open.

3. The intersection of any finite family of open subsets is open.

A pair $(X, \mathcal{T})$ consisting of a set X together with a topology $\mathcal{T}$ on X is called a *topological space*.

A topological *manifold* of dimension n is a topological space M satisfying the following properties:

1. M is a Hausdorff space. For every pair of distinct points, there exist two disjoint open subsets of M that each contain one of the points.
2. M is second-countable. There exists a countable basis for the topology of M.
3. M is locally Euclidean of dimension n. Each point of M has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

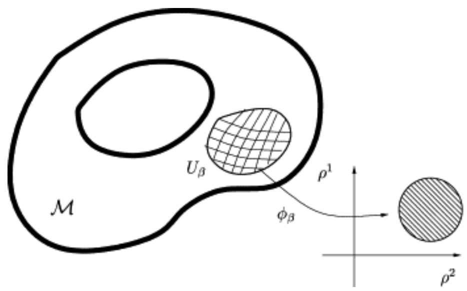

Figure 1: Topological manifold

In differential geometry, it is useful to have a notion of smoothness on these manifolds such that we can perform calculus on them. To do this, we need some more definitions to quantify what "smoothness" means.

First, we define *smooth functions*. If U and V are open subsets of Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, a function $F : U \to V$ is smooth or infinitely differentiable (denoted by C^∞) if each of its component functions has continuous partial derivatives of all orders. If F is also bijective and has a smooth inverse map, it is called a diffeomorphism.

Now we generalize this definition to manifolds. A *coordinate chart* on M is a pair (U, φ) where U is an open subset of M and $\varphi : U \to \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} = \varphi(U) \subseteq \mathbb{R}^n$.

If $(U, \varphi), (V, \psi)$ are two charts such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the *transition map* from φ to ψ. The two charts are *smoothly compatible* if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.

We define an *atlas* for M to be a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible with each other. $\mathcal{A}$ is *maximal* if it is not properly contained in any larger smooth atlas.

If M is a topological manifold, a *smooth structure on* M is a maximal smooth atlas $\mathcal{A}$. A smooth manifold is a pair $(M, \mathcal{A})$, that is, a manifold topology with a smooth structure.

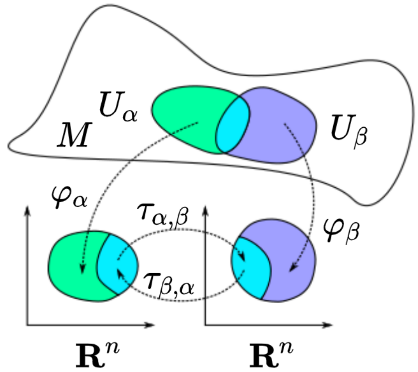

Figure 2: Two charts on a manifold, and their respective transition map

2.2 2-forms

In general, differential forms are mathematical objects that provide a standardized approach to define taking integrals over curves, surfaces, solids, and higher-dimensional manifolds; we will end up integrating them over manifolds to give a measure of their volume. Furthermore, an m-form can be thought of as an oriented density that can be integrated over an m-dimensional oriented manifold.

To make a smooth manifold symplectic, it must be equipped with a special 2-form, a differential form that essentially assigns a scalar to every pair of tangent vectors at each point on the manifold.

Formally, any function $\psi : D \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ satisfying the conditions of

1. linearity in each of its two column-vector variables:

$$\psi(\mathbf{x}, a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = a\psi(\mathbf{x}; \mathbf{u}, \mathbf{w}) + b\psi(\mathbf{x}; \mathbf{v}, \mathbf{w})$$

   and

$$\psi(\mathbf{x}, \mathbf{u}, a\mathbf{v} + b\mathbf{w}) = a\psi(\mathbf{x}; \mathbf{u}, \mathbf{v}) + b\psi(\mathbf{x}; \mathbf{u}, \mathbf{w})$$

2. antisymmetry: $\psi(\mathbf{x}; \mathbf{v}, \mathbf{u}) = -\psi(\mathbf{x}; \mathbf{u}, \mathbf{v})$

is called a *differential 2-form* on a set $D \subseteq \mathbb{R}^m$.

2.3 Symplectic Manifolds

Now, we are ready to define the core concept of symplectic geometry. A *symplectic manifold* is a smooth $2n$-dimensional manifold M along with a differential 2-form $\omega \in \Omega^2(M)$ called the *symplectic form* with the two properties:

1. ω is closed as $d\omega = 0$, i.e. the exterior derivative of ω vanishes.

2. ω is non-degenerate: if $\omega_x(v, w) = 0$ for all $v \in T_xM$, then $\omega = 0$.

Assigning this symplectic form to M is referred to as giving M a *symplectic structure*. Note that T_xM denotes the tangent space of M at a point x, and recall that the tangent space is the set of all possible tangent vectors to curves on M passing through x; we will expand upon tangent spaces later.

Relating this to quantum mechanics, 2-forms help with turning a function (the Hamiltonian representing the total energy of the system) into a flow (the trajectories of the system in phase space, like the solutions to Hamilton's equations). However, a symplectic form specifically does this in a method compatible with Hamilton's equations, which is why it is helpful.

Now we present some well-known examples of symplectic manifolds.

Example: $\mathbb{R}^{2n}$

Let $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is symplectic, and the set

$$\left\{ \left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \ldots, \left(\frac{\partial}{\partial y_n}\right)_p \right\}$$

is a symplectic basis of T_pM.

$\mathbb{R}^{2n}$ is the standard symplectic structure and serves as the prototype for all symplectic manifolds, though it has trivial topology.

Another well-known example of a symplectic manifold is a Kähler manifold, which has three compatible structures: a complex structure, a Riemannian structure, and a symplectic structure. We will omit its formal definition for brevity.

3 Symplectomorphisms

Now we will investigate maps between symplectic manifolds. Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional symplectic manifolds, and let $\varphi : M_1 \to M_2$ be a diffeomorphism. Then φ is a *symplectomorphism* if $\varphi^*\omega_2 = \omega_1$. In other words, a symplectomorphism is a smooth, invertible map between two symplectic manifolds that preserves their symplectic structures, and hence preserves areas.

3.1 Theorem (Darboux)

This theorem is one of the most fundamental results in symplectic geometry.

Theorem: Any symplectic manifold (M^{2n}, ω) is locally symplectomorphic to the "trivial" symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$, where $\omega_0 = \sum_{j=1}^{n} dx^j \wedge dy^j$. In other words, there exists an open neighborhood $U \subset M$ of p and a diffeomorphism $\psi : U \to \mathbb{R}^{2n}$ such that ψ is a symplectomorphism.

A chart $(U, x^1, \ldots, x^n, y^1, \ldots, y^n)$ is called a *Darboux chart*. The existence of Darboux charts means there are no "local invariants" in symplectic geometry (e.g., no analogue of curvature as contrasted to Riemannian geometry). In the context of physics, Darboux coordinates are position/momentum pairs.

The Darboux theorem was first proved by Gaston Darboux in 1882, in connection with his work on ordinary differential equations arising in classical mechanics. A proof discovered in 1971 by Alan Weinstein is based on the theory of time-dependent flows. A more elementary proof given in *Introduction to Smooth Manifolds* by John M. Lee follows the following outline:

1. The smooth coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ on an open subset $U \subseteq M$ are Darboux coordinates if and only if their Poisson brackets satisfy

$$\{x^i, y^j\} = \delta^{ij}; \quad \{x^i, x^j\} = \{y^i, y^j\} = 0.$$

   Note that a Poisson bracket is an important binary operation Hamiltonian mechanics between functions depending on phase space and time, satisfying anticommutativity ($\{f, g\} = -\{g, f\}$), billinearity ($\{af + bg, h\} = a\{f, h\} + b\{g, h\}$, $\{h, af + bg\} = a\{h, f\} + b\{h, g\}$, for $a, b \in \mathbb{R}$), Leibnez's rule ($\{fg, h\} = \{f, h\}g + f\{g, h\}$), and the Jacobi identity ($\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$). In an intuitive sense, a Poisson bracket measures how the flows generated by two functions "twist" relative to the symplectic form ω.

2. Proceeding with induction on k, we can show that for each $k = 0, \ldots, n$, there are smooth functions $(x^1, \ldots, x^k, y^1, \ldots, y^k)$ vanishing at p and satisfying the above in a neighborhood of p such that the $2k$-tuple of 1-forms $(dx^1, \ldots, dx^k, dy^1, \ldots, dy^k)$ is linearly independent at p. Recall that a 1-form is a linear combination of the differentials of the coordinates. When $k = n$, this proves the theorem, and the full Darboux chart has been constructed.

4 Cotangent Bundles

Another important natural example of symplectic manifolds are cotangent bundles. Unlike the previous trivial example of $\mathbb{R}^{2n}$, cotangent bundles encode the topology of an underlying space. Cotangent bundles are defined similarly to tangent bundles, which we shall define first.

Let M be a smooth manifold and $p \in M$. The *tangent space* at p, denoted T_pM, is the vector space of all directional derivatives at p. Its elements are called *tangent vectors*.

The *tangent bundle* TM of a manifold M is the disjoint union of all tangent spaces:

$$TM = \bigsqcup_{p \in M} T_pM.$$

It is naturally a smooth manifold of dimension $2 \dim M$.

Given a vector space V, its *dual space* V^* is the space of all linear functionals $\alpha : V \to \mathbb{R}$. Recall that a *linear functional* is a linear map from a vector space to its field of scalars. If V is finite-dimensional, $\dim V^* = \dim V$.

Next, in a similar vein, the *cotangent space* at p, denoted T_p^*M, is the dual space of T_pM. Its elements are called *cotangent vectors* (or *1-forms* at p).

The *cotangent bundle* T^*M is the disjoint union of all cotangent spaces:

$$T^*M = \bigsqcup_{p \in M} T_p^*M.$$

It is also a smooth manifold of dimension $2 \dim M$, with local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ where (q^i) are coordinates on M and (p_i) are fiber coordinates for 1-forms $\alpha = \sum_{i=1}^n p_i \, dq^i$,.

To understand what fiber coordinates are, we will go on a digression about fiber bundles.

A *fiber bundle* is a quadruple (E, B, π, F) where:

- E (the *total space*) and B (the *base space*) are smooth manifolds,
- $\pi : E \to B$ is a surjective projection map,
- F (the *fiber*) is a manifold, and
- Locally, E is diffeomorphic to the product space $B \times F$ (i.e., for every $p \in B$, there exists a neighborhood $U \subseteq B$ of p such that there exists a homeomorphism between $\pi^{-1}(U)$ and $U \times F$).

Given a fiber bundle $\pi : E \to B$ and its projection map, the *fiber* over a point $p \in B$ is the preimage:

$$E_p := \pi^{-1}(p) \subseteq E.$$

For every $p \in B$, E_p is diffeomorphic to F.

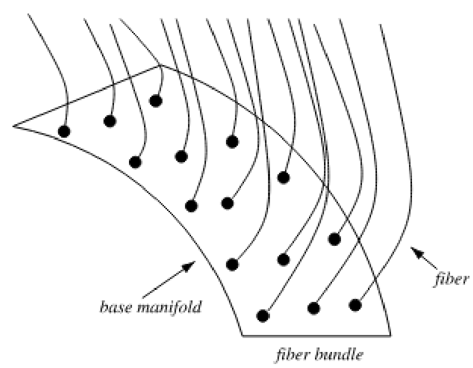

Figure 3: Visualizing a fiber bundle. A common intuitive picture is a hairbrush.

Fiber coordinates label points within each fiber, while base coordinates label points on the base manifold.

Now we can see that cotangent bundles are naturally symplectic manifolds. The base coordinates are analogous to position, and the fiber coordinates are analogous to momentum components.

Theorem: Every cotangent bundle is a symplectic manifold Q with the canonical symplectic form, the Poincare two-form $\omega = \sum_{i=1}^n dp_i \wedge dq^i$, where $(q^1, \ldots, q^n)$ being any local coordinates on Q and $(p_1, \ldots, p_n)$ being fiberwise coordinates with respect to the cotangent vectors $dq^1, \ldots, dq^n$.

5 Lagrangian Submanifolds

Lagrangian submanifolds are an important type of submanifold in symplectic geometry. They generalize solutions to Hamilton's equations where the configuration space is half the phase space dimension, providing a bridge between classical and quantum systems.

Let (M, ω) be a $2n$-dimensional symplectic manifold. A submanifold Y of M is a *Lagrangian submanifold* if, at each $p \in Y$, T_pY is a Lagrangian subspace of T_pM, i.e., $\omega_p|T_pY \equiv 0$ (ω vanishes on L) and $\dim T_pY = \frac{1}{2}\dim T_pM$. Equivalently, if $i : Y \hookrightarrow M$ is the inclusion map, then Y is Lagrangian if and only if $i^*\omega = 0$ and $\dim Y = \frac{1}{2}\dim M$.

5.1 Weinstein's Lagrangian Neighborhood Theorem

Now we provide a theorem emphasizing the usage of these submanifolds.

Theorem: Let M be a smooth $2n$-dimensional manifold with two symplectic forms ω_1 and ω_2 on M. Consider a compact Lagrangian submanifold $i : L \hookrightarrow M$ of both (M, ω_1) and (M, ω_2). Then there exist two open neighborhoods U_1 and U_2 of L in M and a diffeomorphism $f : U_1 \to U_2$ such that $f^*\omega_2 = \omega_1$ and $f|_L = \mathrm{id}_L$.

The theorem states that every Lagrangian submanifold locally resembles the zero section of T_pM with its canonical symplectic structure. It hence links classical states to quantum states via cotangent bundles.

6 References and Further Reading

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Figures taken from Wikipedia.