Lie Groups and Lie Algebras

Anish Sankar

Abstract

In this paper, we introduce the Lie group, the Lie algebra, and the relationship between the two through critical theorems in Lie theory. This will allow us to understand Sophus Lie's theory of continuous groups, bringing differential equations to abstract algebra.

1 Introduction

In the second half of the 19th century, Norwegian mathematician Sophus Lie was studying group actions on manifolds. Specifically, Lie wished to be able to study local behavior of these group actions, implying some sort of differential group theory. Essentially, Lie wished to do for differential equations what Évariste Galois had done for algebraic equations through Galois theory. This led to the creation of the Lie group, a manifold whose points form a group.

Once we have established what a Lie group is, we wish to be able to study it infinitesimally using calculus and differential equations. In order to do so, we introduce the Lie algebra. Intuitively, a Lie algebra is a tangent vector space at the identity of a Lie group. We can still define the structure on its own, so it may seem disconnected from the Lie group we previously defined. However, a theorem which we will cover later will show that any Lie algebra is a tangent space of a Lie group.

We will mainly be focusing on real Lie groups and Lie algebras, but the same structures exist in complex spaces. Thus, we will let $\mathbb{K}$ mean either $\mathbb{R}$ or $\mathbb{C}$, since properties apply in both fields. However, we will be mainly focusing on $\mathbb{R}$, especially for the examples provided.

2 The Lie Group

Definition 1: A *Lie group* is a smooth manifold G whose points act as a group under some smooth binary operation, i.e. there exists some binary operation $\cdot : G \times G \to G$ such that

- $\cdot$ is smooth
- For any $a, b, c \in G$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- There exists some $e \in G$ such that $e \cdot a = a \cdot e = a \ \forall a \in G$
- For all $a \in G$ there exists some point $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$ and the map $a \mapsto a^{-1}$ is smooth

For some common examples of Lie groups, consider the following:

- Euclidean space $\mathbb{R}^n$ under addition
- The circle S^1, or $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ under complex multiplication
- The special orthogonal group $SO(3) = \{X \in GL(3, \mathbb{R}) | X^T X = I, \det X = 1\}$ (a 3-manifold in $\mathbb{R}^9$)

Example (Circle S^1): Consider the circle $S^1 \subset \mathbb{R}^2$, a smooth 1-manifold. As a group, we can consider multiplication given by

$$(x, y) \cdot (u, v) = (ux - vy, uy + vx)$$

and the inverse operation given by

$$(x, y)^{-1} = (x, -y).$$

Both of these operations are smooth, giving a Lie group. As a group, this is isomorphic to $SO(2)$.

Note that even though we work in $\mathbb{R}^n$, we may consider complex multiplication since it is a known group operation. Similarly, we can create a Lie group from S^3 using the quaternions. However, other unit spheres do not have any similar underlying algebraic structure allowing us to form Lie groups.

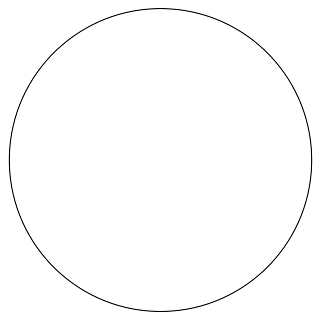

Figure 1: The 1-sphere $S^1 \subset \mathbb{R}^2$

Example (Torus T^2): Consider the torus

$$T^2 = \{((R + r\sin\theta)\cos\phi, (R + r\sin\theta)\sin\phi, r\cos\theta) \in \mathbb{R}^3 | R, r \in \mathbb{R} \text{ constants}, \theta, \phi \in [0, 2\pi)\} \subset \mathbb{R}^3,$$

another smooth manifold. We can describe any point on the torus using the angles θ and ϕ. Thus, let us consider the group operation given by

$$(\theta_1, \phi_1) \cdot (\theta_2, \phi_2) = (\theta_1 + \theta_2, \phi_1 + \phi_2)$$

and inverse map

$$(\theta, \phi)^{-1} = (-\theta, -\phi),$$

where these operations all work mod 2π. We may observe that this group is isomorphic to $(\mathbb{R}/2\pi\mathbb{R}) \times (\mathbb{R}/2\pi\mathbb{R})$ under addition.

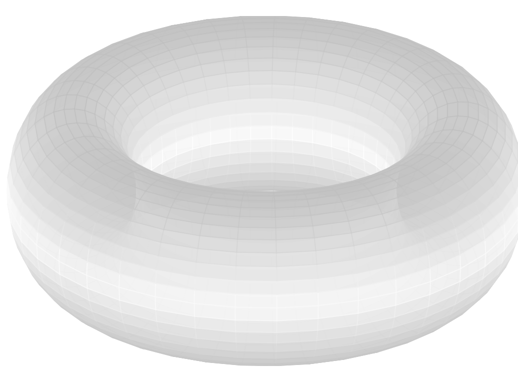

Figure 2: The 2-torus $T^2 \subset \mathbb{R}^3$

3 The Lie Algebra

Definition 2: A *Lie algebra* over $\mathbb{K}$ is a vector space $\mathfrak{g}$ equipped with a bilinear bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket* such that this bracket is antisymmetric ($[X,Y] = -[Y,X]\ \forall X,Y \in \mathfrak{g}$) and satisfies the Jacobi identity ($[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0\ \forall X,Y,Z \in \mathfrak{g}$).

Proposition: The Lie bracket has the alternating property, i.e. $[X,X] = 0\ \forall X \in \mathfrak{g}$.

Proof: Since the Lie bracket is antisymmetric, $[X,X] = -[X,X]$. Thus, we can conclude $[X,X] = 0$.

4 Connecting the Lie Group and Lie Algebra

4.1 The Lie Algebra as a Tangent Space

Consider some Lie group G and its tangent space at the identity T_eG.

Proposition: For any Lie group G, we can define some bracket so that T_eG is a Lie algebra $\mathfrak{g}$.
The proof will be covered throughout this section.

Definition: Let $g,h \in G$ where G is a Lie group. Define the map $L_h : G \to G$ defined by $g \mapsto hg$. We call this map a *left translation*. We can similarly define a *right translation* as $R_h : G \to G$ with $g \mapsto gh$.

These are diffeomorphisms with inverses $L_{h^{-1}}$ and $R_{h^{-1}}$, respectively. Consider the induced map $(L_h)_* : T_gG \to T_{L_h(g)}G$. This map, which is a *pushforward* (conceptually the same idea as taking a derivative) of L_h at a point g, is an isomorphism, since L_h is a diffeomorphism.

Consider a vector field X on G and let X_g be X evaluated at g for some $g \in G$.

Definition: A vector field X is *left invariant* if $X_{hg} = (L_h)_*(X_g)\ \forall g,h \in G$.

Intuitively, a left invariant vector field is a vector field where the induced maps of left translation only shifts the vector field along G instead of altering the values.

From this definition, we can determine that every X_g is determined by $(L_g)_*(X_e)$, so the value of X at the identity determines its value everywhere else in the Lie group.

We can consider the action of a vector field X on some smooth scalar function $f : G \to \mathbb{K}$. We can think of a vector X_g as acting as a directional derivative of f evaluated at g (or in the direction of X_g). We can consider the map $X(f) : G \to \mathbb{K}$ with the mapping $g \mapsto X_g(f)$.

Definition: $\text{Lie}(G)$ is the set of all left invariant vector fields of a Lie group G.

$\text{Lie}(G)$ is a vector space. In fact, since each vector field X is determined entirely by pushforwards of translations of X_e, $\text{Lie}(G)$ is isomorphic to the tangent space T_eG by the mapping from $\text{Lie}(G)$ to T_eG given by $X \mapsto X_e$, giving an isomorphism between vector spaces.

Proposition: The bracket operation given by $[X,Y](f) = X(Y(f)) - Y(X(f))$ defines a Lie bracket.

Proof: Let us check that the necessary conditions are satisfied.
Bilinearity follows since X and Y are linear operations.
Antisymmetry:

$$[Y,X] = Y(X(f)) - X(Y(f)) = -(X(Y(f)) - Y(X(f))) = -[X,Y]$$

Jacobi identity:

$$
\begin{aligned}
&[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y] \\
&=[XY-YX,Z]+[YZ-ZY,X]+[ZX-XZ,Y] \\
&=(XY-YX)Z-Z(XY-YX)+(YZ-ZY)X-X(YZ-ZY)+(ZX-XZ)Y-Y(ZX-XZ) \\
&=XYZ-YXZ-ZXY+ZYX+YZX-ZYX-XYZ+XZY+ZXY-XZY-YZX+YXZ \\
&=0
\end{aligned}
$$

Thus, $[X,Y]=XY-YX$ defines a Lie bracket. This is the Lie bracket commonly used for matrix algebras. Conceptually, this Lie bracket, also called a commutator, quantifies the commutativity (or lack thereof). Returning now to our earlier proposition, we see now that $\mathrm{Lie}(G)\cong T_eG$ with the bracket $[X,Y]=XY-YX$ forms a Lie algebra $\mathfrak{g}$.

Example (Torus T^2): Consider T^2, which we previously showed to be a Lie group. Consider the tangent plane at the identity $(0,0)$. This tangent space spans a plane $\mathbb{R}^2$.

Consider the points $(\theta_1,\phi_1),(\theta_2,\phi_2)\in T^2$. From these, we have a pushforward map

$$(L_{(\theta_1,\phi_1)})_*:T_{(\theta_2,\phi_2)}T^2\to T_{(\theta_1+\theta_2,\phi_1+\phi_2)}T^2$$

In Euclidean space, the pushforward $(L_{(\theta_1,\phi_1)})_*$ is given by the Jacobian of

$$L_{(\theta_1,\phi_1)}(\theta_2\,\phi_2)=(\theta_1+\theta_2,\phi_1+\phi_2),$$

which we can calculate as $\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

Thus, the pushforward is trivial, and our Lie bracket is as well.

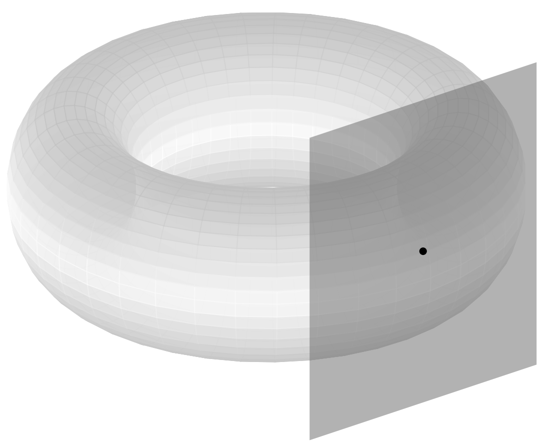

Figure 3: T^2 with a tangent plane

Example (SO_3): Consider the matrix group $SO(3)=\{X\in GL(3,\mathbb{R})|X^TX=I,\det X=1\}$ and the set of matrix-valued curves $\gamma:\mathbb{R}\to SO(3),\gamma(0)=I$ that span the tangent plane at I. All of these γ must, by definition, satisfy $\gamma(0)=I$. From the definition of $SO(3)$, we also know that $\gamma(t)^T\gamma(t)=I$ and $\det\gamma(t)=1\ \forall t$.

From here, since the Lie algebra is the "derivative" of the Lie group, we simply take the derivative of the equation $\gamma(0)^T\gamma(0)=I$ to get

$$\dot\gamma(0)^T\gamma(0)+\gamma(0)^T\dot\gamma(0)=0.$$

Since $\gamma(0)=\gamma(0)^T=I$, the equation becomes

$$\dot\gamma(0)^T=-\dot\gamma(0).$$

Letting $X = \dot{\gamma}(0)$, we have the algebra defined by

$$X^T = -X,$$

which gives us skew-symmetric matrices. Differentiating the second condition gives us $\mathrm{tr}X = 0$, which gives us no new information, since that is a property of skew-symmetric 3×3 matrices. Using the Lie bracket $[X, Y] = XY - YX$, this defines the Lie algebra $\mathfrak{so}_3$.

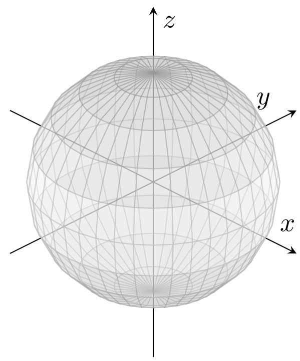

Figure 4: A visualization of $SO(3)$ as rotations in $\mathbb{R}^3$

Proposition: If a Lie group G is Abelian, the Lie bracket of $\mathfrak{g} = \mathrm{Lie}(G)$ is trivial.

Proof: Since $gh = hg \ \forall g, h \in G$, $L_h(g) = R_h(g)$. Therefore, if we have some $m, n \in G$ such that $m = gnh = (L_g \circ R_h)(n)$, we can rewrite this as

$$m = (L_g \circ L_h)(n) = L_{gh}(n)$$

Thus, X_m can be written as $(L_{gh})_*(X_n)$, which simply shifts the vector field and does not alter it. Therefore, any field X is a constant vector field. Since any two fields X and Y are constant, $XY = YX$, so

$$[X, Y] = XY - YX = XY - XY = 0$$

4.2 The Exponential Map

In order to further study the relationship between the two structures, we can define a map from $\mathfrak{g}$ to G.

Definition: A one-parameter subgroup of G is a continuous homomorphism $\varphi : \mathbb{R} \to G$.

Definition: The exponential map $\exp : \mathfrak{g} \to G$ is defined as

$$\exp(X) = \gamma_X(1),$$

where γ_X is a one-parameter subgroup satisfying $\gamma'_X(0) = X$.

4.3 Lie's Third Theorem

Theorem (Lie's Third Theorem): For any finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K}$, there exists a Lie group G such that $\mathfrak{g} = \mathrm{Lie}(G)$.

To prove this, we will use a different theorem, which we will not prove here.

Theorem (Ado's Theorem): Any finite-dimensional Lie algebra $\mathfrak{g}$ over a zero characteristic field $\mathbb{K}$ (such as $\mathbb{R}$ or $\mathbb{C}$) can be mapped to a Lie algebra of square matrices under the commutator bracket as a subalgebra $\mathfrak{g} < \mathrm{End}(\mathbb{K})$.

Ado's theorem, named after Russian mathematician Igor Ado who published the theorem in 1935, allows us to now think of any finite-dimensional Lie algebra as a matrix algebra, or a subset of the general linear algebra $\mathfrak{gl}(n) = \mathrm{Lie}(GL(n))$.

Proof (Lie's Third Theorem): using Ado's theorem, any Lie group $\mathfrak{g}$ can be written as isomorphic to a subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(n)$, which is the Lie algebra of $GL(n)$. We can see that this algebra corresponds to a Lie group generated by $H = \exp(\mathfrak{h}) < GL(n)$. Thus, this Lie group H has Lie algebra $\mathfrak{g}$.

4.4 Baker-Campbell-Hausdorff (BCH) Formula

Baker-Campbell-Hausdorff Formula:

$$\exp X \exp Y = \exp(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y] + ...),$$

where the argument on the right side contains infinitely many further terms with higher orders of Lie brackets.

Russian mathematician Eugene Dynkin provided an explicit formula for all terms in 1947, but we will not cover this since we will not be doing much calculation in this paper, and for what we do, we need not know the coefficients of later terms.

The main idea to notice here is that the BCH formula acts as a measure of non-commutativity. This should make sense, since the difference is based only on the Lie bracket.

Proposition: A Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ with trivial bracket has Abelian Lie group G if the exp map is surjective.

Proof: We know from Lie's Third Theorem that such a group G exists. Using the BCH formula, we see that all the brackets in the right side evaluate to 0, giving

$$\exp X \exp Y = \exp(X + Y).$$

Similarly, we can arrive at

$$\exp Y \exp X = \exp(Y + X),$$

so we have

$$\exp X \exp Y = \exp Y \exp X.$$

Letting $g, h \in G$ be $\exp X$ and $\exp Y$, respectively, we now have $gh = hg$.

We require surjectivity so all of G is included in the exp mapping, but if exp is not surjective, we still have that $\exp(\mathfrak{g}) < G$ is Abelian.

In fact, using this implication, we can conjecture that two Lie groups that are distinct up to isomorphism may have the same Lie algebra. This can occur when one group covers the other. For example, $SU(2) = \{X \in GL(2, \mathbb{C}) | X^\dagger X = I, \det X = 1\}$ is a covering group of $SO(3)$ (note that these groups are not isomorphic), and $\mathfrak{su}(2) = \mathrm{Lie}(SU(2)) \cong \mathrm{Lie}(SO(3)) = \mathfrak{so}(3)$.

5 References

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